This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

600047122M

600047122M

PRACTICAL GEOMETRY,

PERSPECTIVE,

AND

ENGINEERING DRAWING.

BY

GEORGE SYDENHAM CLARKE,

CAPTAIN ROYAL ENGINEERS.

SECOND EDITION.

LONDON:
E. & F. N. SPON, 16, CHARING CROSS.
NEW YORK:
35, MURRAY STREET.
1884.

1837. e. 2

PREFACE.

THE great importance of Descriptive Geometry, not merely as an aid to the Draughtsman but as an instrument of education, has never yet been fully recognised in this country. And yet there are few subjects which call so many mental forces into action, few subjects so well calculated to form a clear, vigorous, and logical mind. A mathematical problem may usually be attacked by what is termed in military parlance the method of "systematic approach," that is to say, its solution may be gradually felt for, even though the successive steps leading to that solution cannot be clearly foreseen. But a Descriptive Geometry problem must be seen through and through before it can be attempted, the entire scope of its conditions as well as each step towards its solution must be grasped by the imagination. It must be "taken by assault."

The want of a text book on the subject has been much felt by the present writer in dealing with large numbers of students. It was found impossible to keep a class of twenty-five fully employed. Time was inevitably lost in explaining away small individual difficulties, and in setting problems suiting individual cases. A text book was required which explained first principles fully and systematically, which preserved a clear and logical sequence throughout its pages, and which furnished examples bearing directly on the subject-matter of each chapter. The time available for this subject was very small, the required text book must contain all that a finished Engineering Draughtsman would require, and no more. Existing works did not satisfactorily meet the wants of the case. Some supposed the student to know too much, others gave him credit for knowing nothing. Some went beyond the requirements of the Engineer, others fell far short of those requirements. Some were mere collections of problems, observing no apparent sequence. The present work is an attempt to embody the Descriptive Geometry course prescribed at Cooper's Hill, and to meet the requirements of young Engineers generally. Failing a thorough knowledge of the principles of the subject, intelligent draughtsmanship is obviously impossible, and those principles, constituting as they do the theory of drawing, must (like all other theory) be studied in the abstract before they can be applied to actual practice. Further, it is obviously desirable that in order to enable the Draughtsman to meet the many and varied requirements of his work, and to prevent him from breaking down in the presence of exceptional conditions, the study of this theory of drawing should be carried somewhat farther than might at first sight appear to be needed. Thus, although the immediate practical bearing of much of the contents of the present work may not be at first apparent, it is nevertheless believed to contain nothing with which every Engineer should not be conversant. The main objects of the writer have been to bring general principles into prominence, to illustrate those principles by a variety of problems fully explained, pointing out at the same time any peculiarities worthy of remark; and finally, to append to each chapter a considerable number of problems, with occasional hints as to their solution. The whole is prefaced by a few remarks on the use of drawing instruments, a few general rules, and (as a help to instructors) a large number of problems in Plane Geometry, classed under various heads. To render the work more complete, and in order that the directly practical element may not be wanting, a final chapter has been added, containing general rules and conventions to be observed in all Engineering Drawings, and also some few hints as to lining, colouring, printing, choice of instruments, &c., which it is hoped will be found of value. Care has been bestowed on the plates, that they may be as far as possible examples of drawing and not mere Euclid figures. By a strange irony, illustrations to works of the kind are too often more fitted to serve as warnings than as guides to the student. The two elements of success in draughtsmanship are sound theory and continued practice. No text book can supply the place of the latter, but due attention to written instructions will at least save time—and paper.

Cooper's Hill.

PREFACE TO SECOND EDITION.

IT is hoped that the new form which has been given to this book will add to such usefulness as it may possess. Although its size has been changed, the original plates are retained, but are now included in a separate cover. The process of turning backwards and forwards when referring to a complicated figure is somewhat confusing, and the plan of keeping text and plates altogether separate—long popular on the Continent—is believed to be the most generally convenient. To render the work more complete and self-contained, a chapter on perspective has been added, in which the *rationale* of the subject is treated with somewhat more fulness than is universal. Other additions and amplifications have been made, which a larger experience, both in teaching and examination, could not fail to suggest.

LONDON, 1883.

CONTENTS.

PLANE GEOMETRY.

CHAPTER I.

PAGE

Drawing Instruments. Miscellaneous Examples. Lines. Lines and Circles. Plane Figures. Parabola. Ellipse. Hyperbola. Cycloidal Curves. Spirals. Miscellaneous Curves 2

PRACTICAL SOLID GEOMETRY.

CHAPTER II.

ORTHOGRAPHIC PROJECTION.

Points and Lines. Elementary Principles and Definitions. Problems. Examples 15

CHAPTER III.

LINES AND PLANES.

First Principles. Planes in various Positions. Problems. Projection of Plane Polygons. Projection of Circles. Examples 25

CHAPTER IV.

THE USE OF INDICES.

PAGE
Principles and Advantages of the Method. Problems. Examples 44

CHAPTER V.

THE PROJECTION OF SOLIDS.

Regular Solids. Surfaces of Revolution. Ruled Surfaces. Conical, Cylindrical, and Helicoidal Surfaces. Projection of various Solids in various Positions. Projection of Screw Threads. Examples 52

CHAPTER VI.

SECTIONS OF SOLIDS BY PLANES.

Sections by Vertical, Horizontal, and Oblique Planes. Sections of Curved Surfaces by Planes. Examples 70

CHAPTER VII.

INTERPENETRATION OF SOLIDS.

Solids contained by Planes; by Curved Surfaces. Tangents to Curves of Double Curvature. Development. Examples .. 81

CHAPTER VIII.

TANGENT PLANES TO SURFACES.

Problems. Examples 93

CHAPTER IX.

DETERMINATION OF SHADOWS.

Outlines of Shadows. Conventional Direction of Rays of Light. Shaded Portions of Solids of Revolution. Examples 101

CHAPTER X.

SHADE-LINES AND SHADING.

PAGE
Shade-lines. Rules. Shading. Principles and Rules 114

CHAPTER XI.

ISOMETRIC PROJECTION.

General Principles. Isometric Scales 119

PERSPECTIVE.

CHAPTER XII.

General Principles. Perspective by simple Orthographic Projection. Perspective by Vanishing Points. Use of Measuring Points 124

ENGINEERING DRAWINGS.

CHAPTER XIII.

General Considerations and Rules. Pencilling. Lining. Shade-lines. Colour. Table of Conventional Colours. Flat Washes. Shading Machinery. Colouring Sections. Printing 165

CHAPTER XIV.

SELECTION OF DRAWING INSTRUMENTS, ETC.

Needle Points. Bows. Dividers. Napier Compasses. Proportional Compasses. Drawing Pens. Sharpening Drawing Pens. Dotting Pens. Parallel Rulers 173

PRACTICAL GEOMETRY.

PLANE GEOMETRY.

INTRODUCTION.

Objects of Practical Plane Geometry.

PRACTICAL PLANE GEOMETRY has two main objects, viz.:

1st. The solution of problems of all kinds by graphic methods, i. e. by pure construction.

2nd. The delineation of the various figures and curves which occur in the Arts and are required for the purposes of engineering drawing generally.

Considered in the first sense, Practical Plane Geometry presents a wide field of operations and one which is being gradually extended by French and German Geometricians so as to comprehend a large number of statical and dynamical problems before treated by analytical methods solely. On the other hand, the constant recurrence of purely geometrical forms in every branch of engineering drawing, and in that of machinery in particular, renders a knowledge of the subject essential to the practical draughtsman.

The present work being intended mainly as a textbook of "Practical Solid" or "Descriptive" Geometry and Perspective, it has been thought advisable merely to propound a considerable number of plane geometrical problems, and to leave them to be worked by the student with the assistance and at the discretion of an instructor. A few remarks on drawing instruments and a few rules of universal application are given, as likely to be of value to beginners.

CHAPTER I.

PLANE GEOMETRY.

Instruments.

THE following instruments are necessary for geometrical drawing:—

Drawing pens.

Two drawing pens, one for fine and the other for thick lines. Great care is required in their use and treatment. They should be held steadily and at a constant angle to the paper. The ink must be introduced between the nibs by means of a small brush or the feather of a quill pen, and as soon as it begins to thicken the points should be wiped clean with blotting-paper previous to refilling. After use, drawing pens should be carefully wiped with wash-leather.

Large compasses.

Two pairs of compasses, one smaller for use as "dividers" only, in transferring lengths from scales, setting off distances, &c.; the other larger, with a movable limb which can be replaced at pleasure by an ink or pencil leg. These latter compasses should have needle points. In using dividers care must be taken not to make holes in the paper, a very slight indent being all that is necessary to mark a point.

Small compasses.

Two small compasses, one having a fixed pencil and the other a fixed pen leg. These are required for drawing small pencil and ink circles. Both should have needle points and jointed limbs. In drawing circles the jointed limbs of the compasses should be bent so as to stand at right angles to the paper.

Spring bows.

Spring dividers and bows will also be found useful when small distances have to be accurately transferred to the paper, or very small pencil and ink circles to be drawn. The distance between the points of these spring dividers and bows is regulated by a screw, which prevents them from spreading, keeps them fixed at any required radius, and allows of an easier and more exact adjustment to that radius.

Set squares. Two large right-angled triangles, called "set squares," made of pearwood or ebonite, one having an angle of 60°, the other of 45°. These are used in drawing lines parallel, perpendicular, and at 30°, 45°, and 60° to each other.

T square. A "T square," suited to the size of the drawing board; one of 30 inches, to be used with an "imperial" (30″ × 22″) board, will be found most generally useful.

Protractor. An ivory "protractor" for setting off angles of any given number of degrees is necessary. This also combines diagonal scales of quarter inches and half inches decimally divided, a scale of chords, and a variety of other useful scales.

Sector. An ivory sector, combining double scales of lines, chords, sines, tangents, secants, &c., will be found useful in facilitating a variety of operations, by saving the trouble of construction.

Pencils. Pencils of the class H H are best suited for mechanical drawing generally, and for geometrical drawing proper, a fine point is much to be preferred to a chisel-shaped one.

Colours, &c. Colours, saucers, Indian ink, and brushes are indispensable.

Paper. Whatman's "cold pressed" is recommended.

General Rules. 1. Make every construction as large as the limits of your paper allow, and bestow all possible care on the pencilling, remembering that in no case is it likely that pencil errors will be obviated in the after inking in.

2. In drawing a line through a point be sure that it does actually pass through the point; small errors are multiplied in any long construction, and may materially affect results.

3. Draw no more lines than are absolutely necessary: a large number of lines tend to confuse a drawing and lead to mistakes in inking.

17. Find the angle between the lines $x + y\sqrt{3} = 0$, and $x - y\sqrt{3} = 2$.

18. Find the length of the perpendicular from the point 1, −2, on the line $x + y - 4 = 0$.

19. Draw the lines represented by the equation $y^2 - 4xy + 3x^2 = 0$.

LINES AND CIRCLES.

Lines and circles.

20. Draw the arc of a circle passing through three given points without using the centre.

(Points 5″, 3·75″, and 2·5″ apart.)

21. Draw the tangent at any point to a circular arc; the centre being supposed unavailable.

(Arc as in No. 20.)

22. Draw two tangents to a given circle from an outside point.

(Circle 1·25″ rs., pt. 3″ from cr.)

23. Draw tangents to two given circles.

(Circles 1·25″ and ·75″ rii., crs. 3″ apart.)

24. Between two given lines, not parallel, draw a series of circles touching each other and the lines.

25. Draw a continuous curve formed of circular arcs, passing through any number of given points.

26. On a given line describe the segment of a circle containing a given angle.

(Line 2″, angle 55°.)

27. Draw a circle touching two given circles and passing through a given point.

(Circles 1·25″ and ·75″ rii., crs. 2·5″ apart.)

28. Draw a circle passing through a given point and touching two given lines not parallel.

29. Draw a circle touching two given circles, one of them in a given point.

(Data as in No. 27.)

30. Draw circles touching three given unequal circles.

(Circles 1″, 1·25″, 1·55″ rii., centres at the angles of an equilateral triangle of 3″ side.)

31. Draw a circle touching a given circle and a given line and passing through a given point external to both.

(Circle 1″ rs., line 1·68″ from cr.)

32. Draw a circle touching a given circle and line, the latter in a given point.

(Data as in No. 31.)

33. Draw a circle touching a given circle and line, the former in a given point.

(Data as in No. 31.)

34. Draw a circle passing through two given points and touching a given line.

(Points 1″ and 1·5″ from line and 2″ apart.)

35. Draw a circle passing through two given points and cutting a given circle in the extremities of a diameter.

(Circle 1·5″ rs., pts. 1·3″ and 1·75″ from cr. and 2·5″ apart.)

36. Draw a series of circles touching a given line, a given circle, and each other.

(Circle 1·85″ rs., line 2·85″ from cr.)

37. Draw a circle of given radius passing through a given point and touching a given circle.

(Circle 1·5″ rs., given rs. 1″.)

38. Draw a circle of given radius touching two given circles.

(1st. Given rs. 1″, circles 1·3″ and ·75″ rii., crs. 2·6″ apart.
2nd. „ 3″, „ „ „ „)

39. Draw a circle of given radius touching a given line and a given circle.

(Circle 1·75″ rs., line 2·75″ from cr., given rs. 1·5″.)

40. Inscribe within a given circle three equal semicircles having their diameters adjacent.

(Circle 2″ rs.)

41. From two given external points draw two lines meeting in a point of the circumference of a given circle and making equal angles with the tangent at that point.

(Circle 2″ rs., pts. 4″ and 3·5″ from cr. and 4″ apart.)

42. Chords are drawn through a given point to a given circle, and tangents are drawn at the extremities of each chord. Shew that the locus of intersection of these tangents is a straight line.

(Circle 2″ rs., pt. 3″ from cr.)

43. Determine the intersection of the circle $y^2 + x^2 = 16$ with the lines $y + x = 1$, $y + x = -3$, $3y + 4x = -15$ (unit ·5″).

PLANE FIGURES.

Plane figures.

44. Construct a triangle, given its perimeter and the ratio of its angles.

(Perimeter 7·9″; angles as 2 : 3 : 4.)

45. Construct a triangle, given one side A B, the altitude and the angle C.

(A B 4″, altitude 2·8″, angle C 60°.)

46. Construct a triangle, given the radius of its inscribed circle and two of its angles.

(Rs. of circle 1·3″, angles 50° and 65°.)

47. Construct a triangle, given its perimeter, altitude, and vertical angle.

(Perimeter 7·5″, altitude 2·2″, vertl. angle 65°.)

48. Construct an equilateral triangle of given area.

(Area 2·4 sqre. inches.)

49. Construct a triangle, given the area and the ratio of its angles.

(Area 2·5 sqre. inches, angles as 2 : 3 : 4.)

50. Reduce an irregular figure to a triangle.

(Figure 8 sides, no side to be less than 1″.)

51. Construct any right-angled triangle having a given perimeter.

(Perimeter 7·5″.)

52. Divide an irregular figure into any number of equal parts by lines drawn through one of its angles.

(Figure 9 sides, no side less than 1″, 10 equal parts.)

53. Divide a triangle into two equal parts by a line parallel to one side.

(Triangle 3″, 2·5″, 3·7″ sides.)

54. Divide a triangle into two equal parts by a line perpendicular to one side.

(Data as in No. 53, line perpr. to longest side.)

55. Divide a triangle into two equal parts by a line passing through a given external point.

(Triangle as in No. 53, point 1·5″ from each extremity of shortest side.)

56. Inscribe any regular polygon in a given circle.

(Rs. of given circle 1·75″, polygon a nonagon.)

57. Construct a regular polygon of given side.

(Side 1·5″, polygon a heptagon.)

58. Construct a regular polygon equal to an irregular four-sided figure.

(Sides of figure 1·75″, 3·5″, 2·5″, 3·3″, diagonal 4·25″, polygon a pentagon.)

59. Inscribe a square in a trapezium having equal adjacent sides.

(Sides 1·5″ and 2·5″, diagonal 2·5″.)

60. Construct a triangle equal to, and having its base on the same line as, a given triangle, but with its vertex in a given point.

(Triangle equilateral, 2·75″ side.)

61. Reduce an irregular four-sided figure to a square of equal area.

(Figure as in No. 58.)

62. Construct an octagon by cutting off the angles of a square.

(Square 2·5″ side.)

63. Inscribe a square in a given triangle.

(Triangle 2″, 3″, and 3·6″ sides.)

64. Construct a parallelogram equal to a given triangle in area and perimeter.

(Triangle as in No. 63, parallelogram to have an angle of 60°.)

65. Divide a circle into any number of parts, equal in area and perimeter. (Rs. of circle, 2″, 5 parts.)

66. Construct a rectilinear figure, whose area shall have a given ratio to that of a given rectilinear figure; the figures to be *similar*.

67. Construct a square equal to a regular polygon.

(Polygon a hexagon of 2″ side.)

68. The side and diagonal of a square are together equal to 6″. Construct the square.

69. Inscribe in a given circle a rectangle of maximum area.

(Circle 2″ rs.)

70. Inscribe three circles in a given isosceles triangle.

(Triangle 3″, 4″, and 4″ sides.)

71. Describe a circle whose area shall have a given ratio to that of a given circle. (Rs. of given circle 1·3″; ratio 2 : 5.)

72. Describe a circle equal in area to the difference of two given circles. (Rii. of circles 2·5″ and 1″.)

73. Describe a circle equal to the sum of two given circles.

(Rii. of circles ·75″ and ·5″.)

74. Construct a regular polygon equal to a given regular polygon, but with one side more.

(Polygon, heptagon 1·3″ side.)

THE PARABOLA.

The parabola.

75. Draw a parabola, given the focus and directrix.

(Focus 2″ from directrix.)

76. Draw a parabola, given the ordinate to the axis, at any point, the tangent at this point, and one other point on the curve.

(Ordinate and tangent inclined at 60°, point 2″ from tangent and 1·5″ from ordinate.)

77. Determine a tangent and normal at any point to a parabola.

(Parabola as in No. 75.)

78. Draw a parabola touching a given circle in a given point, and such that its axis touches the circle in another given point.

(Circle 1·5″ rs., pts. at the extremities of a chord 3·75″ long.)

79. If a triangle be inscribed in a parabola, shew that the points in which the sides produced cut the tangents at the opposite angles are in the same straight line.

(Parabola as in No. 75.)

80. Draw a parabola having given three points on the curve and the *direction* of its axis.

THE ELLIPSE.

The ellipse.

81. Draw an ellipse, given the major and minor axes.
(Axes 4″ and 2·5″.)

82. Draw an ellipse, given two conjugate diameters.
(Diameters 4″ and 3″, inclined at 55°.)

83. Draw a tangent and normal at any point to an ellipse.
(Ellipses as in Nos. 81 and 82.)

84. Draw tangents from any external point to an ellipse.

85. Given the conjugate diameters of an ellipse, determine the axes.
(Diameters as in No. 82.)

86. Determine the centre and radius of curvature at any point of an ellipse, and draw the evolute of one quadrant.
(Ellipse as in No. 81.)

87. Given a diameter and one ordinate of an ellipse, determine the conjugate diameter.

88. Inscribe an ellipse in a given trapezium.
(Trapezium 2″, 2·2″, 4″, and 4·5″ sides, diagonal 3·5″.)

89. Draw a curve formed of circular arcs with five centres, approximating to a half ellipse.
(Span 4″, versine 1·5″.)

90. Given the foci, describe an ellipse touching a given line.
(Foci 3″ apart.)

91. A diameter of an ellipse, parallel to the tangent at any point, meets the focal distances of the point. From the points of intersection lines are drawn perpendicular to the focal distances. Shew that these perpendiculars intersect in the minor axis.
(Axes of ellipse 5″ and 3″.)

THE HYPERBOLA.

The hyperbola.

92. Draw a hyperbola, given the vertex and foci.
(Foci ·75″ and 2·25″ from vertex.)

93. Draw a rectangular hyperbola, given the centre and the vertex.
(Vertex 2″ from centre.)

94. Draw a tangent and normal at any point, to a hyperbola.
(Hyperbola as in No. 92.)

95. Draw a tangent to a hyperbola from any point outside the curve.
(Hyperbola as in No. 92.)

96. Determine the asymptotes of a hyperbola whose vertex and foci are given, and draw the conjugate hyperbola.
(Given hyperbola as in No. 92.)

CYCLOIDAL CURVES.

Cycloidal curves.

97. Draw a cycloid.

(Rs. of generating circle 1″.)

98. Determine the tangent and normal to a cycloid at any point.

(Cycloid as in No. 97.)

99. Draw the trochoid described by a point within the generating circle.

(Rs. of generating circle 1·25″, pt. ·75″ from its centre.)

100. Draw the trochoid described by a point without the generating circle.

(Rs. of generating circle 1″, pt. 1·5″ from centre.)

101. Draw the epicycloid.

(Rs. of directing circle 2·5″, rs. of generating circle 1″.)

102. Draw the hypocycloid.

(Data as in No. 101.)

103. Shew that the hypocycloid becomes a straight line when the diameter of the generator is half that of the director.

(Rs. of director 2″.)

104. Shew that the hypotrochoid becomes an ellipse when the diameter of the generating circle is half that of the director.

(Rs. of director 2″.)

SPIRALS.

Spirals.

105. Draw the involute of a given circle to pass through a given point.

(Rs. of circle 1·5″, pt. 3·4″ from cr.)

106. Draw the spiral, the distance of any point of which from the pole is directly proportional to the angle made by the radius with the axis. (Spiral of Archimedes.)

(The spiral to cut the axis after one convolution at a point 1·5″ from the pole.)

107. Draw the spiral, the distance of any point of which from the pole is proportional to the square of the angle made by the radius with the axis.

(The curve to cut the axis after one convolution at a point ·75″ from the pole.)

108. Draw the spiral, the radius at any point of which varies inversely as the angle it makes with a fixed axis. (The Reciprocal spiral.)

(Asymptote 5″ from pole.)

109. Draw the Equiangular or Logarithmic spiral.
(Greatest diameter 5″, pole 3″ from the extremity of the diameter.)

110. Draw the Ionic Volute.
(Greatest radius 3·25″.)

MISCELLANEOUS.

Miscellaneous.

111. Draw the Companion to the cycloid.
(Radius of circle 1″.)

112. Draw the Harmonic curve.
(Height 1·5″.)

113. Draw the Lemniscate, given the length and height.
(Length 3″, height 1″.)

114. Draw the Cissoid.
(Pole 2″ from asymptote.)

115. Draw the superior and inferior Conchoid.
(Pole 1″ from asymptote. Portion of radius vector intercepted between the curve and its asymptote 1·25″.)

PRACTICAL SOLID GEOMETRY.

INTRODUCTION.

Objects of Descriptive Geometry.

THE main object of "Practical Solid" or "Descriptive" Geometry is "the investigation of methods by which bodies of three dimensions of various forms and in various positions can be correctly represented on paper." Of such methods two only—"Orthographic Projection" and "Isometric Projection"—are as a rule employed by the Engineer draughtsman, while "Perspective" or "Radial Projection" more especially concerns the Architect.

Besides its directly practical uses, Descriptive Geometry affords a means of correctly representing various complex curved surfaces, and of "deducing and verifying a variety of theorems resulting from their forms and relative positions." Descriptive Geometry is in this sense the servant of Analytical Geometry of three dimensions. The subject frequently presents considerable difficulties to beginners. These arise chiefly from a want of imagination. The single plane of the paper has often to do duty for several planes variously inclined to it, and it is not easy at first to imagine the latter in their true relative positions. The student is particularly advised to obtain a complete mastery of first principles and to accustom his mind to the realisation of the actual positions of points and lines only, before proceeding further. It may be added, that a thorough knowledge of the elements of Descriptive Geometry is absolutely essential to intelligent draughtsmanship. A mere unreasoning acquaintance with a few general rules is sure to prove insufficient in the presence of any slightly exceptional conditions.

ORTHOGRAPHIC PROJECTION.

CHAPTER II.

POINTS AND LINES.

Theorem 1.

THE position of a point in space is determined if its situation with respect to two fixed planes is known.

The two fixed planes are assumed horizontal and vertical respectively, and the point is referred to these planes by dropping from it a perpendicular on each of them.

Definitions. Planes of projection. Projectors. Projections.

The two planes are termed "Co-ordinate Planes," or "Planes of Projection"; the perpendiculars dropped on them from any point, "Projectors," and the intersection of these perpendiculars with the planes, "Projections" of that point.

Plan and elevation.

Further, the intersection of the perpendicular with the horizontal plane (i. e. the horizontal projection) is termed the "Plan," and the intersection of the perpendicular with the vertical plane (i. e. the vertical projection) is termed the "Elevation" of the point.

Ground line. Dihedral angles.

The intersection of the planes of projection is called the "Ground Line," or xy, and the four right angles formed by these intersecting planes are spoken of as "Dihedral Angles."

The above definitions should be clearly understood and carefully remembered.

Notation.

A clear notation is essential to render a complicated construction intelligible, and to prevent confusion arising between plan and elevation. It is usual to designate a point in space by a large alphabetical letter, e. g. A, denoting its plan by the corresponding small letter *a*, and its elevation by *a'*. A point given by its projections *a*, *a'* would therefore be spoken of, either as the point A, or as the point *a a'*. A second and third elevation of the same point would consistently be denoted by *a''*, *a'''*.

Fig. 1, Pl. I. Fig. 1, Pl. I., gives an *end view* of the two planes of projection; four points, A, B, C, D, are situate one in each of the dihedral angles, and following the above notation their elevations are lettered *a'*, *b'*, *c'*, *d'*, and their plans *a*, *b*, *c*, *d*. But inasmuch as the paper presents only one available plane, we must suppose the vertical plane to rotate into coincidence with the horizontal plane, carrying with it the elevations *a'*, *b'*, *c'*, *d'*. The arrows and arcs in Fig. 1 shew the direction in which this rotation is supposed to take place.

Fig. 2, Pl. I. In Fig. 2 we have the representation of the same points, A, B, C, D, when the planes of projection have been brought into coincidence. This figure should be clearly understood, as it forms the basis of the whole subject of Orthographic Projection. The student is therefore recommended before going further to bend up a piece of paper or card, so as to form two planes at right angles, and then, after marking the projections of an imaginary point, to turn the paper down into one plane again. Fig. 2 also shews the use of a notation distinguishing plan and elevation in enabling us to see at a glance in which of the four dihedral angles any given point lies. The point A is, as may be seen by referring to Fig. 1, in the first angle, that is to say, *above* the horizontal and *in front of* the vertical plane; the point B is in the second angle, i. e. *above* the horizontal and *behind* the vertical plane; while C is in the third angle, i. e. *below* the horizontal and *behind* the vertical plane; and D is in the fourth angle, i. e. *below* the horizontal and *in front of* the vertical plane. The following important theorems may now be noticed.

Theorem 2. The plan and elevation of any point must always lie in

a line at right angles to the ground line; e.g. *a a'* (Fig. 2).

Theorem 3. The distance of the plan of any point from the ground line is always equal to the distance of the point from the vertical plane; e.g. *a a* (Fig. 2) is equal to *a* O (Fig. 1), and therefore equal to A*a'* (Fig. 1), and A*a'* is the distance of A from the vertical plane.

Theorem 4. The distance of the elevation of any point from the ground line is equal to the height of the point above the horizontal plane; e.g. *a' a* (Fig. 2) is equal to *a'* O (Fig. 1), and therefore to A*a*, and A*a* is the height of the point A above the horizontal plane.

Theorem 5. All points *in* the vertical plane have their plans in the ground line; e.g. the point E (Fig. 2).

Theorem 6. All points *in* the horizontal plane have elevations in the ground line; e.g. the point G (Fig. 2).

Theorem 7. When a point is *in* the ground line the plan and elevation coincide; e.g. the point H (Fig 2).

If the plan and the elevation of any point on one vertical plane is given, any number of elevaticns on different vertical planes can be readily deduced.

Thus, in Fig. 2, the line Z Z is assumed as the ground line of a new vertical plane, and the elevations (a'', b'', c'', &c.) of the original points on this second plane are determined. To effect this it is merely necessary to draw perpendiculars from the plans of all the points to the new ground line, and then set off the distances $a'\,a$, $b'\,\beta$, &c., from this ground line on these perpendiculars. These distances must of course be set off on the same side of the ground line as the plans or not, according as the points are below or above the horizontal plane.

LINES.

Projections of a line. The Projections of a straight line are made up of the projections of all its points, and as the projectors of all points on a straight line must be contained by planes at right angles to the planes of projection and passing through the line, the intersection of these perpendicular denoting its plan by the corresponding small letter *a*, and its elevation by *a′*. A point given by its projections *a*, *a′* would therefore be spoken of, either as the point A, or as the point *a a′*. A second and third elevation of the same point would consistently be denoted by *a″*, *a‴*.

Fig. 1, Pl. I. Fig. 1, Pl. I., gives an *end view* of the two planes of projection; four points, A, B, C, D, are situate one in each of the dihedral angles, and following the above notation their elevations are lettered *a′*, *b′*, *c′*, *d′*, and their plans *a*, *b*, *c*, *d*. But inasmuch as the paper presents only one available plane, we must suppose the vertical plane to rotate into coincidence with the horizontal plane, carrying with it the elevations *a′*, *b′*, *c′*, *d′*. The arrows and arcs in Fig. 1 shew the direction in which this rotation is supposed to take place.

Fig. 2, Pl. I. In Fig. 2 we have the representation of the same points, A, B, C, D, when the planes of projection have been brought into coincidence. This figure should be clearly understood, as it forms the basis of the whole subject of Orthographic Projection. The student is therefore recommended before going further to bend up a piece of paper or card, so as to form two planes at right angles, and then, after marking the projections of an imaginary point, to turn the paper down into one plane again. Fig. 2 also shews the use of a notation distinguishing plan and elevation in enabling us to see at a glance in which of the four dihedral angles any given point lies. The point A is, as may be seen by referring to Fig. 1, in the first angle, that is to say, *above* the horizontal and *in front of* the vertical plane; the point B is in the second angle, i. e. *above* the horizontal and *behind* the vertical plane; while C is in the third angle, i. e. *below* the horizontal and *behind* the vertical plane; and D is in the fourth angle, i. e. *below* the horizontal and *in front of* the vertical plane. The following important theorems may now be noticed.

Theorem 2. The plan and elevation of any point must always lie in

a line at right angles to the ground line; e.g. *a a′* (Fig. 2).

The distance of the plan of any point from the ground line is always equal to the distance of the point from the vertical plane; e.g. *a a* (Fig. 2) is equal to *a* O (Fig. 1), and therefore equal to A *a′* (Fig. 1), and A *a′* is the distance of A from the vertical plane. **Theorem 3.**

The distance of the elevation of any point from the ground line is equal to the height of the point above the horizontal plane; e.g. *a′ a* (Fig. 2) is equal to *a′* O (Fig. 1), and therefore to A *a*, and A *a* is the height of the point A above the horizontal plane. **Theorem 4.**

All points *in* the vertical plane have their plans in the ground line; e.g. the point E (Fig. 2). **Theorem 5.**

All points *in* the horizontal plane have elevations in the ground line; e.g. the point G (Fig. 2). **Theorem 6.**

When a point is *in* the ground line the plan and elevation coincide; e.g. the point H (Fig 2). **Theorem 7.**

If the plan and the elevation of any point on one vertical plane is given, any number of elevaticns on different vertical planes can be readily deduced.

Thus, in Fig. 2, the line Z Z is assumed as the ground line of a new vertical plane, and the elevations (a'', b'', c'', &c.) of the original points on this second plane are determined. To effect this it is merely necessary to draw perpendiculars from the plans of all the points to the new ground line, and then set off the distances *a′ a*, *b′ β*, &c., from this ground line on these perpendiculars. These distances must of course be set off on the same side of the ground line as the plans or not, according as the points are below or above the horizontal plane.

LINES.

The Projections of a straight line are made up of the projections of all its points, and as the projectors of all points on a straight line must be contained by planes at right angles to the planes of projection and passing through the line, the intersection of these perpendicular **Projections of a line.**

denoting its plan by the corresponding small letter *a*, and its elevation by *a'*. A point given by its projections *a*, *a'* would therefore be spoken of, either as the point A, or as the point *a a'*. A second and third elevation of the same point would consistently be denoted by *a''*, *a'''*.

Fig. 1, Pl. I. Fig. 1, Pl. I., gives an *end view* of the two planes of projection; four points, A, B, C, D, are situate one in each of the dihedral angles, and following the above notation their elevations are lettered *a'*, *b'*, *c'*, *d'*, and their plans *a*, *b*, *c*, *d*. But inasmuch as the paper presents only one available plane, we must suppose the vertical plane to rotate into coincidence with the horizontal plane, carrying with it the elevations *a'*, *b'*, *c'*, *d'*. The arrows and arcs in Fig. 1 shew the direction in which this rotation is supposed to take place.

Fig. 2, Pl. I. In Fig. 2 we have the representation of the same points, A, B, C, D, when the planes of projection have been brought into coincidence. This figure should be clearly understood, as it forms the basis of the whole subject of Orthographic Projection. The student is therefore recommended before going further to bend up a piece of paper or card, so as to form two planes at right angles, and then, after marking the projections of an imaginary point, to turn the paper down into one plane again. Fig. 2 also shews the use of a notation distinguishing plan and elevation in enabling us to see at a glance in which of the four dihedral angles any given point lies. The point A is, as may be seen by referring to Fig. 1, in the first angle, that is to say, *above* the horizontal and *in front of* the vertical plane; the point B is in the second angle, i. e. *above* the horizontal and *behind* the vertical plane; while C is in the third angle, i. e. *below* the horizontal and *behind* the vertical plane; and D is in the fourth angle, i. e. *below* the horizontal and *in front of* the vertical plane. The following important theorems may now be noticed.

Theorem 2. The plan and elevation of any point must always lie in

a line at right angles to the ground line; e.g. ***a a'*** (Fig. 2).

The distance of the plan of any point from the ground line is always equal to the distance of the point from the vertical plane; e.g. ***a a*** (Fig. 2) is equal to ***a*** O (Fig. 1), and therefore equal to A *a'* (Fig. 1), and A *a'* is the distance of A from the vertical plane. **Theorem 3.**

The distance of the elevation of any point from the ground line is equal to the height of the point above the horizontal plane; e.g. *a'* ***a*** (Fig. 2) is equal to *a'* O (Fig. 1), and therefore to A *a*, and A *a* is the height of the point A above the horizontal plane. **Theorem 4.**

All points *in* the vertical plane have their plans in the ground line; e.g. the point E (Fig. 2). **Theorem 5.**

All points *in* the horizontal plane have elevations in the ground line; e.g. the point G (Fig. 2). **Theorem 6.**

When a point is *in* the ground line the plan and elevation coincide; e.g. the point H (Fig 2). **Theorem 7.**

If the plan and the elevation of any point on one vertical plane is given, any number of elevaticns on different vertical planes can be readily deduced.

Thus, in Fig. 2, the line Z Z is assumed as the ground line of a new vertical plane, and the elevations (a'', b'', c'', &c.) of the original points on this second plane are determined. To effect this it is merely necessary to draw perpendiculars from the plans of all the points to the new ground line, and then set off the distances *a' a*, *b' β*, &c., from this ground line on these perpendiculars. These distances must of course be set off on the same side of the ground line as the plans or not, according as the points are below or above the horizontal plane.

LINES.

The Projections of a straight line are made up of the projections of all its points, and as the projectors of all points on a straight line must be contained by planes at right angles to the planes of projection and passing through the line, the intersection of these perpendicular **Projections of a line.**

planes with the planes of projection must be the projections of the line. Hence, the projections of all straight lines are straight lines.

Projecting plane. The perpendicular planes containing the projectors of all points on a line are termed the *projecting planes* of the line. In the case of a curved line the projecting plane is replaced by a curved surface, termed the *projecting surface* of the curve.

All lines parallel to neither plane of projection must meet both.

Traces of a line. The point in which a line cuts the vertical plane of projection is termed the *vertical trace*, and that in which it cuts the horizontal plane the *horizontal trace* of the line.

A line parallel to one of the planes of projection has therefore a trace on the other.

A line parallel to both planes of projection has no trace.

A line parallel to neither plane of projection has a trace on each.

The trace of a line on a plane to which it is perpendicular coincides with its projection on that plane.

A little thought will render the following theorems evident.

Theorem 8. A line parallel to both planes of projection has both its projections parallel to the ground line.

Theorem 9. A line parallel to one plane of projection has its projection on the other plane parallel to the ground line.

Theorem 10. A line at right angles to one plane of projection has for its projection on that plane a point, and on the other a line perpendicular to the ground line.

Theorem 11. A line passing through the ground line has its projections meeting in a point of the ground line.

Theorem 12. The projection of a finite line on any parallel plane is an equal line.

Theorem 13. The projection of a finite line on any plane oblique to its direction is less than the line, and decreases as the inclination of the line to the plane increases; in fact, if L be the length of any line in space, l its projection on any plane, and θ the inclination of the line to the plane, then $l = L \cos. \theta$.

The projection of any plane angle on a parallel plane is an equal angle. Theorem 14.

The projections on any plane of equal parallel lines are themselves equal and parallel. Theorem 15.

A line is determined if the position of two points on it are known; hence it may be referred to either as the line *a b*, *a' b'*, or A B, consistently with the notation adopted on *p.* 15. Notation.

Fig. 3, Pl. I., shews the projections of lines in six different positions with respect to the planes of projection. Fig. 3, Pl. I.

A line parallel to both planes. Case I.

A line parallel to the vertical and inclined to the horizontal plane. Case II.

A line inclined to the vertical and parallel to the horizontal plane. Case III.

A line perpendicular to the vertical and parallel to the horizontal plane. Case IV.

A line parallel to the vertical and perpendicular to the horizontal plane. Case V.

A line passing through the ground line and inclined to both planes of projection. Case VI.

Figs. 4 and 5, Pl. II., shew lines *a b*, *a' b'* in two different positions inclined to both planes of projection. Figs. 4, 5, Pl. II.

Given the projections of a line, to determine its traces. **Problem I.**

The horizontal trace of a line is by definition (*p.* 18) a point in the horizontal plane, hence its elevation must be on the ground line (Theor. 6, *p.* 17). But as the horizontal trace is a point on the line itself, its elevation must be on the elevation of the line. Therefore the point in which the elevation of the line cuts the ground line must be the elevation of the horizontal trace; a line drawn from this point perpendicular to the ground line and cutting the *plan* will give this trace. By similar reasoning it is evident that the vertical trace is determined by producing the *plan* to meet the ground line, and drawing a line from this point perpendicular to the ground line and cutting the *elevation*.

Figs. 4, 5, Pl. II.

The horizontal and vertical traces of the lines given in Figs. 4 and 5, Pl. II., are determined by this construction and lettered $h\,t$ and $v\,t$ respectively.

Problem II. **Given the projections of a line, to determine the real length of the segment between any two points on it, and also its inclination to the horizontal and vertical plane.**

Fig. 6, Pl. II.

Fig. 7, Pl. II.

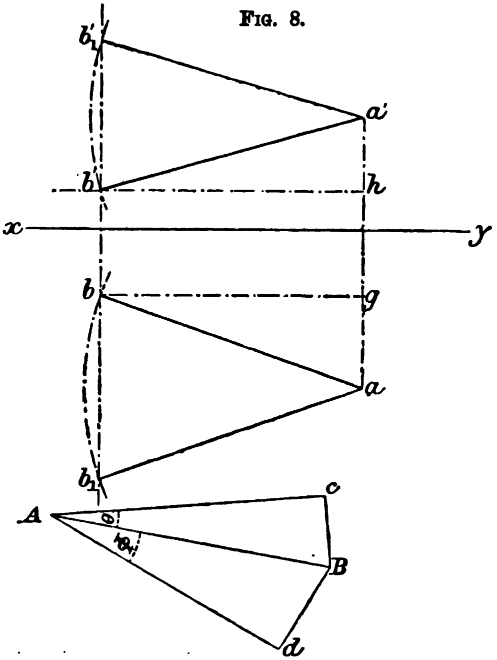
A and B, given by their projections $a\,a'$, $b\,b'$, are the two points. Now the real line A B, its plan $a\,b$, and the projectors of the points A and B form a trapezium in space, of the form shewn in Fig. 7. In this trapezium, the side $a b$ is the plan of the line A B; A a is the projector of A and therefore equal to $a'\,a$ (Theor. 4), and similarly B b is equal to $b'\,\beta$ (Fig. 6). Now if in this trapezium a line B h be supposed to be drawn parallel to $a b$, the angle θ will evidently be the angle made by the line A B with the horizontal plane, and the solution of the right-angled triangle A B h will give the length A B and the angle θ. But in this triangle the base h B is equal to $a\,b$, and the side A h is equal to the difference between A a and B b, or the difference between the heights of the points A and B in space. This difference is equal to the line $a'\,h'$ (Fig. 6). Hence, if in Fig. 6 a line a A is drawn at right angles to $a\,b$ and equal to $a'\,h'$, then A and b being joined give the angle θ, while the line A b is equal to A B in space. The trapezium A $a\,b$ B was of course merely alluded to for explanation, all the necessary construction being given in Fig. 6. The construction for finding the inclination of A B to the vertical plane will be of course exactly similar, the line b' B at right angles to $a'\,b'$ being made equal to $a\,g$ (the difference between the vertical projectors of A and B), ϕ will be the inclination of A B to the vertical plane. It should be noticed that a' B and A b are both equal to the real length of the line A B.

The student is recommended to master the above problem thoroughly, as it involves several important elementary principles.

Given the length AB of a finite line, and also its inclinations to the planes of projection, to determine its plan and elevation. **Problem III.**

This problem is merely the converse of Prob. II. Suppose *a a′* (Fig. 8) to be the point from which the line is to be drawn. This point must be assumed, as it in nowise enters into the conditions of the problem, for the *lengths* of the plan and elevation of a line and its angles of inclination remain unchanged if the line is moved in any direction parallel to itself (Theor. 15). Let θ and ϕ be the angles the line is to make with the horizontal and vertical planes of projection respectively. Anywhere on the paper construct two right-angled triangles AB_1c Fig. 8.

FIG. 8.



A B d, both having the given distance A B as hypothenuse, while one (*A B c*) has an angle θ, and the other (*A B d*) has an angle ϕ. Now the distance *B c* must be equal to the *difference* of the heights of the two ends of the required line, that is to the difference of the heights of the elevations of the two ends of the line above the ground line (Theor. 4, *p.* 17). Make a' *h* equal to B *c*, and draw a line through *h* parallel to the ground line. Then the elevation of the other end of the line required must be on the line drawn through *h*. But in the triangle A B *d* the side A *d* must be equal to the length of the elevation of the required line. With a' as centre, and radius equal to A *d*, describe an arc cutting the horizontal through *h* in b'. Then $a' b'$ is the elevation of the required line. But in the triangle *A B c*, the side A *c* must be equal to the length of the plan of the required line. With *a* as centre, and radius equal to A *c*, describe an arc cutting the line projected down from b' in *b*. Then *a b* is the plan of the required line.

Note *a*. If a line parallel to the ground line be drawn from *b* to cut the line $a a'$ in *g*, then it will be found that *a g* is equal to *B c*.

Note *b*. The point b' might have been taken at a height *above* a' equal to *B d* instead of below. In this case we should have obtained the elevation $a' b'_1$. Similarly, the arc from *a* as centre, with radius equal to A *d*, will cut the line projected down from b' in another point b_1, as well as in *b*. Hence the line might have another plan $a\ b_1$. Since each plan may have either of the two elevations and still fulfil the given conditions, it is evident that four lines can be drawn from the point $a a'$ *to the left*, which will satisfy the condition of the problem. Four other lines can be drawn from the point $a a'$ to the right. Hence the problem admits of eight solutions. On the other hand, four of the lines which can thus be drawn coincide *in direction* with the other four, and therefore there are only four lines which can be drawn from a given point to make given angles with the planes of projection.

Theorem 16. The *sum* of the inclinations of a line to two planes at right angles to each other can never exceed 90°, and it attains this maximum when the line lies in a plane perpendicular to both planes.

Note *c*. In this case the traces must both lie on a line at right angles to the ground line.

Draw a line through a given point parallel to a given line. **Problem IV.**

The projections of the line required will be parallel to those of the given line (Theor. 15) and must pass through those of the given point. In Fig. 4, Pl. II., a line is drawn through pp' parallel to the line A B. Fig. 4, Pl. II.

Determine the angle contained between two lines. **Problem V.**

[N.B. If both the lines are parallel to one of the planes of projection, then the angle contained by their projections on this plane will be the angle required (Theor. 14).]

In Fig. 9, Pl. II., $a\,b, a'\,b'$ and $c\,d, c'\,d'$ are two lines intersecting in $i\,i'$. Determine their horizontal traces $h\,t_1$, $h\,t_2$ (Prob. I.). Then evidently the angle $h\,t_1$, I, $h\,t_2$, is the angle required and $h\,t_1$, i, $h\,t_2$, is its plan. To determine this angle, join $h\,t_1$, $h\,t_2$, and suppose the triangle $h\,t_1$, I, $h\,t_2$, to rotate down into the horizontal plane about this line. The point I will evidently come down into the horizontal plane on the prolongation of the perpendicular $i\,p$ drawn from i to the line of rotation $h\,t_1$, $h\,t_2$; to determine I therefore we have only to determine the real length of the line of which $i\,p$ is the plan. The height of I is given by its elevation i' (Theor. 4), and is equal to $i'\,a$. Making $i\,i''$ at right angles to $i\,p$ equal to $i'\,a$, we obtain $i''p$, which is the real length of $i\,p$. Produce $p\,i$ to I, make $p\,$I equal to $p\,i''$; then joining I, $h\,t_1$, and I, $h\,t_2$, the angle $h\,t_1$, I, $h\,t_2$ (θ), is the required angle contained by the lines A B, C D.

EXAMPLES.

1. Determine the projections of points in the following positions with respect to the planes of projection.

a. 2″ above horizontal plane; 1·5″ in front of ve[illegible]lane.
b. 1·5″ below ,, 2″ ,,
c. 2·25″ above ,, 1·75″ behind [illegible]
d. 1·75″ below ,, 2·25″ in front of [illegible]
e. 1·75″ above horizontal plane; behind verti[illegible] from ground line.
f. Below horizontal plane; 1·75″ in front of ve[illegible] 2·75″ from $x\,y$. (Theors. 2, 3, 4, and Fig. 2.)

2. Two points are situated 2″ and 1″ from both planes of projection respectively, their plans are 3″ apart. Determine the length of the line joining them and the angles it makes with the planes of projection. (Theor. 13 and Prob. II.)

3. Assume the projections of a point 2″ from the horizontal and 2·5″ from the vertical plane, and draw from this point a line 4″ in length meeting the ground line. (Theor. 13.)

4. Assume a point 2·5″ from the horizontal and 2″ from the vertical plane, and draw from it a line 4″ long inclined at 20° and 35° to those planes respectively. (Theor. 13.)

5. A line is inclined at 25° and 40° to the horizontal and vertical planes respectively, its traces are 4″ apart. Determine its projections. (Theor. 13, Prob. I.)

6. An equilateral triangle of 2″ side is the plan of a triangle whose angles are 1″, 2″, 3″, respectively above the horizontal plane. Determine an elevation of this triangle on a vertical plane, making 40° with the plan of one of its sides. Determine also the real form of the triangle. (At starting draw the equilateral triangle so that one side makes 40° with *x y*. Then apply Theor. 4 and Prob. II.)

7. A parallelogram (sides 2·5″ and 2″, shorter inclined 30° to *x y*) is the plan of a rhombus of 3″ side. One angle of this rhombus is 2″ above the horizontal plane. Draw the elevation of the rhombus, and deduce a second elevation on a new ground line inclined 40° to *x y*. (Theors. 13 and 4.)

8. Assume the projections of any three points unequally distant from the planes of projection and forming a triangle in space. Determine a fourth point, in the same plane with the three assumed points and 2″ and 3″ from the horizontal and vertical planes respectively. (On each of two sides, produced if necessary, of the triangle obtained by joining the three assumed points, determine a point 2″ from the horizontal plane. Theor. 4. On the line joining the two points thus obtained, determine a point 3″ from the vertical plane. Theor. 3.)

9. Two lines inclined to the horizontal plane at angles of 20° and 35° respectively are drawn from a point situated 2″ from both planes of projection. The plans of these lines are inclined at 115° to each other. Determine the real angle included by the lines. (Prob. V.)

10. The plan and elevation of a line are inclined at 35° and 45° respectively to *x y*. The horizontal trace of the line is 2″ from the vertical plane. Draw a second line inclined at 30° and making 70° with the first. (Converse of Prob. V.)

11. Given the projections of three points and the plan of a fourth point. Determine the elevation of the latter so that it shall lie in the same plane as the other three.

CHAPTER III.

LINES AND PLANES.

THE lines in which a plane cuts the vertical and horizontal planes of projection are respectively termed the "vertical" and "horizontal traces" of that plane, and since evidently no two planes can have the same traces, the position of a plane in space is fully determined if its traces are known. The rotation of the vertical plane of projection about the ground line into coincidence with the horizontal plane is supposed to take place as explained in Chap. II., and illustrated by Fig. 1, Pl. I. **Traces of a plane. Theorem 17.**

By bending a piece of cardboard to represent the planes of projection, and using a set square to represent planes in any required position with respect to the cardboard planes, the following important Theorems will be rendered evident:—

A horizontal plane can have only one trace, and this trace will be parallel to the ground line. **Theorem 18.**

A vertical plane parallel to the vertical plane of projection can have only one trace, and this trace also will be parallel to the ground line. **Theorem 19.**

A vertical plane *not* parallel to the vertical plane of projection will have two traces, of which one (the vertical trace) must be perpendicular to the ground line. **Theorem 20.**

A plane perpendicular to the vertical plane of projection but not horizontal will have two traces, of which one (the horizontal trace) must be perpendicular to the ground line. **Theorem 21.**

The two traces of a plane perpendicular to *both* planes of projection will coincide in a line at right angles to the ground line. **Theorem 22.**

All other planes will have both a vertical and a hori- **Theorem 23.**

zontal trace, and these two traces, *unless both parallel to the ground line,* must meet in a point on it. Conversely, any two lines which meet in a point on the ground line must be the traces of some plane.

Note. The two traces of a plane may as a special case coincide in a line making an angle with the ground line, and conversely any line drawn at any angle to the ground line will represent the traces of a plane. In this case the angles made by the plane with the planes of projection are equal.

Theorem 24. The traces of a plane contain the traces of all lines lying in that plane; since all lines in the plane, if they cut the planes of projection at all, can only cut it in points lying on the lines in which the plane itself cuts the planes of projection.

Theorem 25. The traces of parallel planes are respectively parallel. This follows from Euc. XI., 16.

Theorem 26. If a plane is perpendicular to either plane of projection, its trace on that plane of projection makes with the ground line an angle equal to the dihedral angle contained between the plane itself and the other plane of projection. Thus, if two lines are drawn from any point of the ground line, one in the horizontal plane at right angles to the ground line and the other in the vertical plane, making θ with the ground line, these lines will be the traces of a plane at right angles to the vertical plane of projection and inclined θ to the horizontal plane of projection.

Theorem 27 If a plane is at right angles to two other planes which are not parallel, it will be at right angles to their intersection (Euc. XI., 19), and it will cut the two planes in two lines, one in each plane. These two lines (both of which will be at right angles to the intersection of the planes) contain an angle equal to the dihedral angle contained by the planes themselves (Euc. XI., def. 6).

Theorem 28. If a line is perpendicular to a plane, its projections will be at right angles to the traces of the plane, each to each;

i. e. the elevation at right angles to the vertical, and the plan to the horizontal trace.

For if a line is perpendicular to a given plane, all planes containing it will be perpendicular to that plane; hence, the "projecting plane" (*p.* 18) of the line will be perpendicular to that plane. But the projecting plane must also be perpendicular to the plane of projection. Therefore the projecting plane is perpendicular to the intersection of the given plane and the plane of projection, that is to the trace of the given plane; while the projection of the line which lies in the projecting plane must also be at right angles to the trace. A card model will render this clear.

In future, the angle made by a plane or line with the *horizontal* plane of projection will be spoken of simply as the *inclination* of that plane or line. Note.

In Fig. 13, Pl. III., are represented, in order, the several cases of planes referred to in Theors. 18, 19, 20, 21, 22 (*p.* 25). Fig. 13, Pl. III.

Fig. 14 shews the general case of a plane oblique to both planes of projection, and also the projections (*a b*, *a′ b′*) of a line lying in that plane (Theor. 24). Fig. 14, Pl. III.

Cases III. and IV., Fig. 13, will also serve to illustrate Theor. 26 (*p.* 26).

If a right cone has its base on either plane of projection, the traces of all planes touching its surface must be tangent to its base, and the inclination of all these planes to that plane of projection will be equal to the angle at the base of the cone. Theorem 29.

All lines which can be drawn from a fixed point to make a given angle with a fixed plane, must lie on the surface of a right cone, of which the vertex is the fixed point, and the angle at the base is equal to the given angle of inclination of the lines to the plane. Theorem 30.

Determine a plane to contain a given point and be parallel to a given plane. **Problem I.**

The traces of the required plane will be parallel to those of the given plane (Theor. 25), and if a horizontal line be drawn through the point parallel to the given plane, it must lie in the plane required; its vertical trace will therefore be a point on the vertical trace of the required plane. Having determined this point, the vertical trace of the required plane must be drawn through it and parallel to that of the given plane, and the horizontal trace through the intersection of the vertical trace with the ground line (Theor. 23) and parallel to the given horizontal trace.

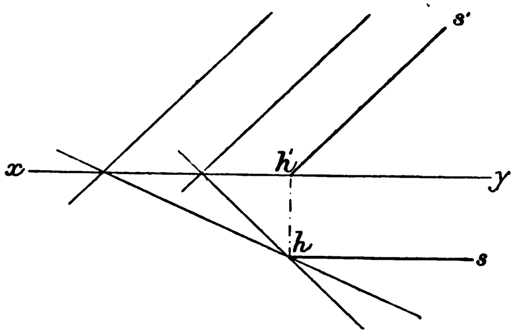
Problem II. **Given the traces of two intersecting planes, to determine the projections of their intersection.**

The traces of this intersection must lie in the traces of both the planes (Theor. 24), since the intersection is a line common to both.

Fig. 15, Pl. III. The point $v\,t$ (Fig. 15, Pl. III.) is therefore the vertical and $h\,t$ the horizontal trace of the intersection, and dropping perpendiculars on the ground line from these points, $v\,t$, β, is the elevation, and $h\,t$, a, the plan required.

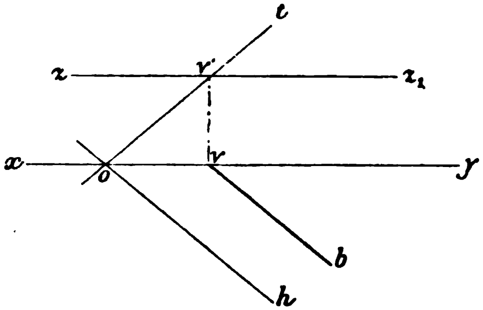
Note *a*. Fig. 16. If two of the traces (the vertical traces for example) are parallel as in Fig. 16, then $h'\,s'$ the elevation of the intersection will be parallel to the vertical traces, and the plan $h\,s$ will be parallel to the ground line.

FIG. 16.



If one of the planes $z z_1$ (Fig. 17) is horizontal, the other plane $t o h$ must cut the plane $z z_1$ and the horizontal plane of projection in parallel lines. (Euc. XI., 16.) Hence the intersection of the planes $z z_1$ and $t o h$ must be parallel to $o h$, and consequently horizontal. But v' (the intersection of $z z_1$ and $t o$) must be its vertical trace. Hence **Note b.** **Fig. 17.**

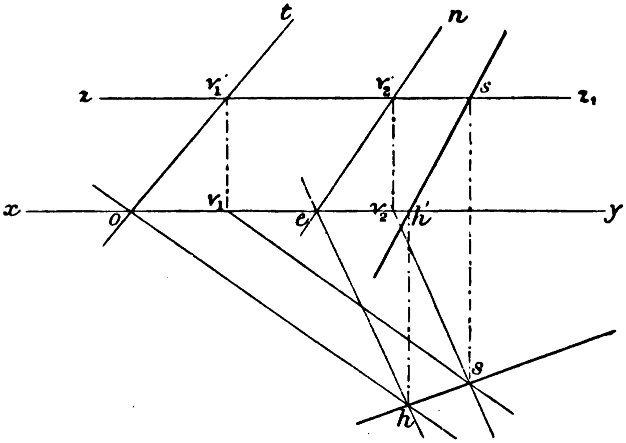
FIG. 17.



$v b$ drawn from v, the plan of this vertical trace, parallel to $o h$, is the plan of the intersection, and $v' z_1$ is its elevation.

Again, the intersection of two of the traces, the vertical traces for example, as in Fig. 18, may be too far off to be conveniently used. **Note c.** **Fig. 18.**

FIG. 18.



In this case, cut both the given planes $t o h$, $n e h$ by any horizontal plane $z z_1$, and obtain $v_1 s$, $v_2 s$ the plans of the intersection of the given planes with $z z_1$ as in the preceding case. Now if any two planes are cut by a third plane, the intersections of those planes with the third plane must intersect in a point on the intersections of the two planes themselves. Hence *s* must be a point on the plan of the intersection of the planes *t o h*, *n e h*. Therefore *h s* is the plan of the intersection of those planes. But *h′* the elevation of *h* is on the ground line, and *s′* the elevation of *s* is on $z z_1$. Join *h′*, *s′*. Then *h′ s′* is the elevation of the intersection of the two given planes.

Problem III. **Determine the angles made by a given plane with the plane of projection.**

Fig. 19, Pl. III.

In Fig. 19, *v′* O *h* is the given plane, and *v′ v t* is a vertical plane perpendicular to O *h* (Theors. 20 and 28). The plane *v′ v t* cuts the plane *v′* O *h* in the line *v′ t*; the lines *t v′* and *t v* therefore contain an angle equal to the required inclination of the plane *v′* O *h*, and we have merely to solve the right-angled triangle *t v v′*. This is done by making *v* T equal to *v t*, and joining T *v′*: the angle *v* T *v′* (a) is therefore the required *inclination*. The construction for determining the lines *s h′*, *h′ h*, and for solving the triangle *h s h′* is exactly similar, and the angle *h* S *h′* (β) is the required inclination of the plane *v′* O *h* to the vertical plane of projection.

Note *a*. It will be evident from the above construction that if the traces of the given plane make equal angles with *x y*, the plane must make equal angles with the planes of projection.

Note *b*. The special case in which both traces coincide in a line inclined to *x y* should give rise to no difficulty.

Problem IV. **Determine the traces of a plane that makes angles θ and ϕ with the horizontal and vertical planes of projection respectively.**

Suppose that two right cones are obtained whose vertices are situated in the horizontal and vertical planes, and bases in the vertical and horizontal planes respectively, and whose generatrices make θ and ϕ with their bases respectively. The bases must be circles one on each plane of projection. Then the required plane must touch both these cones, and its traces must be tangent to their bases (Theor. 29). As a necessary condition that these cones can have a common tangent plane, they must envelope a common sphere, to which the plane will also be tangent. This sphere must evidently have its centre on the ground line.

Fig. 20, Pl. III.

Commence, therefore, by assuming a sphere whose centre v (Fig. 20) is a point in the ground line.

The circle $a\,b\,c$ will be both plan and elevation of this sphere. The two lines $v'\,e$, $h\,f$ touching the sphere and inclined θ and ϕ to the ground line will be generatrices, and v' and h the vertices of the two cones. Arcs of circles (centre v and radii $v\,e$, $v\,f$) in the horizontal and vertical plane respectively are portions of the bases of the two cones, and the lines $h\,o$, $v'\,o$ touching these arcs will be the traces of the plane required. (Theor. 29, *p*. 27.)

Note *a*.

The *sum* of the given inclinations $(\theta + \phi)$ must not exceed 180°, nor be less than 90°. When $\theta + \phi = 180°$ the plane will be at right angles to both planes of projection, its traces will therefore coincide in a line at right angles to $x\,y$ (Case V., Fig. 13, Pl. III.). When $\theta + \phi = 90°$ both traces will be parallel to $x\,y$.

Note *b*.

To determine a plane to contain a given point and make given angles with the planes of projection, it will be necessary merely to draw *any* plane inclined at the given angles, and to determine a parallel plane containing the given point (Prob. I.).

Problem V.

Determine the dihedral angle contained between two given planes.

The dihedral angle is measured by the angle contained between two lines, one in each plane, perpendicular to their line of intersection and drawn from any point on it. (Euc. XI., def. 6.)

This angle is determined by drawing a third plane perpendicular to both the given planes and therefore to their intersection; the lines in which this third plane cuts the given planes will then contain the required angle (Theor. 27, *p*. 26).

Fig. 21, Pl. IV.

In Fig. 21, Pl. IV., $v'\,A\,t$, $v'\,B\,t$ are the traces of the two given planes. Their intersection $v\,t$, $v'\,t'$ is determined (Prob. II.). The line $p\,q$, drawn at right angles to $v\,t$ the plan of this intersection, will be the horizontal trace of some plane perpendicular to V T

(Theor. 28, *p.* 26), and therefore to both the given planes (Theor. 27). Now, if I is the point in which this perpendicular plane cuts V T, the angle *p* I *q* in space is the angle required. The triangle *p* I *q* is solved if we know I *s* the perpendicular drawn from I on the base *p q*. To obtain I *s*, the vertical triangle *v′ v t* (which contains I *s*) is rotated into the vertical plane about *v′ v*, carrying with it the point *s*. This rotation is shewn by arcs, S and T being the positions of *s* and *t* after rotation. Then a perpendicular from S on *v′* T determines S I′, the length required. The triangle *p* I *q* is solved by rotation about *p q*, and *s* I being made equal to S I′, *p* I *q* is the angle required. It may be remarked that the angle *v′* T *v* is the *inclination* of the intersection of the two given planes.

Problem VI. **Determine the intersection of a given line and plane.**

If any second plane containing the given line be determined, the required point will lie on the intersection of this second plane with the given plane.

Fig. 22, Pl. IV. In Fig. 22, Pl. IV., *v′* B *t* is the given plane; *a b, a′ b′* the given line; for simplicity, the second or auxiliary plane is assumed vertical, so that *v′ v, v b* are its traces. The intersection (*v t, v′ t′*) of this vertical auxiliary plane with the given plane is now determined (Prob. II.). The required point must be common to the two lines *a′ b′*, *a b* and *v′ t′*, *v t*; hence *i′* is its elevation, and *i* its plan.

Problem VII. **Determine the angle which a given line makes with a given plane.**

If from any point on the line a perpendicular is drawn to the plane (Theor. 28, *p.* 26), and its intersection with the plane determined (Prob. VI.); then the line joining this intersection and the intersection of the given line and plane will make with the first line the angle required (Euc. XI., def. 5). This angle can be determined by Prob. V., Chap. II. The problem may also be solved by determining the angle which the perpendicular makes with the given line, to which the required angle is evidently complementary.

Given the plan of a point lying in a given plane to determine: **Problem VIII.**

1st. The elevation of the point.

2nd. The lines drawn from the point at a given inclination and in the given plane.

Suppose p to be the given plan, t' A s the given plane, and θ the given inclination (Fig. 23, Pl. IV.). **Fig. 23, Pl. IV.**

1st. The required elevation of the point must lie in the elevation of a horizontal line passing through the point and lying in the plane; of this line $p\,t$ is the plan, t' the vertical trace and $t'\,p'$ the elevation. The required elevation of the point is therefore p'.

2nd. The height of the point P is $p'\,\pi$ (Theor. 4, *p.* 17), and constructing the right-angled triangle $p'\,n\,\pi$ with the given base-angle θ, $n\,\pi$ is evidently the ***length in plan*** of all lines which can be drawn from P inclined θ to the horizontal plane.

Since the required lines are to lie in the plane t' A s, their horizontal traces will be found by cutting the trace A s by arcs from centre p with radius $n\,\pi$ (Theor. 24 *p.* 26); f and s will therefore be the horizontal traces of the lines required, and $f'\,s'$ the elevations of these horizontal traces. The required lines are therefore $p'\,f'$, $p\,f$, and $p'\,s'$, $p\,s$.

The converse of the first part of the above construction serves to determine the *plan* of a point in a given plane if the elevation of that point is given or assumed. **Note *a*.**

If the given inclination of the lines is *less* than that of the plane, there will always be two lines fulfilling the condition. If this given inclination is equal to that of the plane, there will be but one line. If the given inclination is greater than that of the plane, the problem is impossible. **Note *b*.**

Through a given point draw a line parallel to a given plane and having a given inclination. **Problem IX.**

Since it is easier to draw a line parallel to a given *line* than to a given *plane*, assume the projections of a point *in* the given plane by means of the first part of Prob. VIII. From this point draw a line having a given inclination and *in* the given plane (Prob. VIII.). Then a line from the given point parallel to the last drawn line (Theor. 15, *p*. 19) will obviously fulfil the conditions of the problem.

Problem X. **Determine the plane containing three given points.**

By joining the points three lines are obtained, each of which lies in the plane required; the traces of two of these lines will therefore determine the required traces of the plane (Prob. I., Chap. II., and Theor. 24, *p*. 26).

Problem XI. **Determine the plane containing a given line and a given point.**

Draw a line through the given point parallel to the given line (Theor. 15, *p*. 19). Determine the traces of this last drawn line and of the given line (Prob. I. Chap. II.). Then the traces of the required plane must be drawn though the traces of the lines (Theor. 24, *p*. 26).

Problem XII. **From a given point draw a line which shall meet two given lines not parallel.**

The line required must evidently be common to two planes, each of which contains the given point and *one* of the given lines. Determine these two planes by Prob. XI., then their intersection (Prob. II.) is the line required.

Problem XIII. **Determine a plane containing one given line and parallel to another given line.**

If from any point in the line which is to be contained by the plane, a line is drawn parallel to the line to which the plane is to be parallel; then the required plane must contain this last drawn line. Hence it remains merely to determine a plane containing two lines which meet (Prob. X., Theor. 24, *p*. 26).

Note. The following problems have special reference to the projection of solids, and should therefore be thoroughly mastered before the latter subject is attempted.

It will be noticed that in the preceding problems given planes have been invariably assumed as *oblique to both* planes of projection. This has been done to render the problems as general as possible. In the following cases, however, given planes will be assumed as perpendicular to the vertical plane of projection, in order to simplify constructions and lessen the number of lines necessary.

This assumption must not be regarded as in any way involving merely special cases of the problems; for it must be remembered that it is always legitimate and not unfrequently desirable to change the vertical co-ordinate plane by assuming a new ground line. Then an elevation on this new ground line being obtained, a second elevation on the original ground line is readily deduced (Chap. II., First Principles).

Problem XIV.

Determine the projections of any polygon, having given the inclination of its plane and also that of one of its sides or diagonals.

Fig. 24, Pl. V.

Suppose the given figure to be a hexagon A B C D E F (Fig. 24, Pl. V.), the inclination of its plane α, and of A F one of its sides β. Having assumed a ground line $x\,y$, draw the traces (O M O N) of a plane at right angles to the vertical plane, and inclined at α (Theor. 26, *p.* 26). In this plane (M O N) assume $a'\,a$ as the projections of the point A of the hexagon.

Note. If any plane is at right angles to the vertical plane of projection, the elevations of all points in it must lie in its vertical trace.

From $a'\,a$ draw a line in the plane M O N and inclined at an angle β (Prob. VIII.). The plan of this line will be $a\,\pi$ equal to $t\,s$, and its elevation will coincide with M O the vertical trace of the plane. Now suppose the plane M O N (carrying with it the point A) to rotate about its horizontal trace till it coincides with the plane of the paper. Drop the perpendicular $a\,r$ from a to the horizontal trace of the plane, then the point A must, when the rotation is completed, be in this perpendicular produced.

But since the real distance of A from the horizontal trace N O is a' O, we obtain the point A turned down by setting off this distance from r along $r\,a$ produced. Now π the horizontal trace of the line $a\,\pi$, being *in* the line N O, is unaffected by the rotation, hence A π is the line

$a\,\pi$ turned down into the plane of the paper. The hexagon A....F can now be described, A F on A π turned down, being made equal to one of its sides. It remains merely to turn the points A....F back into the plane M O N. This operation, the converse of that described in turning the point A down, is shewn by arcs and lines.

A second elevation a''....f'' of the hexagon A....F on a second vertical plane $y\,z$ is also shewn, and is obtained by the application of Theor. 4, *p.* 17.

Note *a.* The above construction is applicable to *any* plane rectilinear figure, whether the line whose inclination is given is a side, a diagonal, or any line whatever in the plane of the figure. The line which is to have the given inclination is first obtained and turned down into the plane of the paper. The figure can then be constructed of its real form on that line and rotated back into the given inclined plane.

Note *b.* This problem is of course subject to the limitation stated in the note to Prob. VIII.

Note *c.* If A is the area of any plane figure, a the area of its projection on any plane, and θ the inclination of its plane to the plane on which it is projected, then $a = \text{A} \cos. \theta$. This is merely an extension of Theor. 13, *p.* 18.

Note *d.* The process by which the plane M O N (Fig. 24) is turned down about its horizontal trace N O should be clearly understood, as it is one which frequently occurs both in determining the real forms of plane figures from their plans and elevations, and also in the converse problem of determining their plans from their real forms. The process is sometimes spoken of as "constructing" a plane, and the lines or points lying in it. It should be noticed that whenever a plane is constructed or turned down about its horizontal trace, any "constructed" or turned-down point in it will always lie in the line drawn from the plan of that point perpendicular to the horizontal trace of the constructed plane; as for example the point A in the perpendicular $a\ r$ (Fig. 24). Further, the plan of any line and the same line constructed, or turned down, will always meet, if produced, in the horizontal trace of the constructed plane; as the lines $a\,b$, A B; $a\,c$, A C; $d\,f$, D F (Fig. 24).

"Construction" of a point, line, or plane.

Note *e.* The symmetry of the plan and elevation of this hexagon should be noticed. This symmetry illustrates the Theorem, equal parallel lines have equal parallel projections (Theor. 15, *p.* 19).

Problem XV. **Determine the plan of any polygon, having given the inclination of any two of its lines.**

Suppose the polygon to be a pentagon A...E, one side

of which (A B) is to be inclined α and one diagonal (B E) β.

A ... E (Fig. 25, Pl. V.) is the pentagon supposed turned down into the plane of the paper. Assume the height of the point B above the plane of the paper, and with B as centre and the assumed height of B as radius, describe a circle. Touching this circle and making angles of α and β respectively, with B A and B E produced, draw the lines $p\,t$, $q\,s$; p and q being their points of contact with the circle. Then t B p, s B q, are right-angled triangles, and B p being equal to the height of B and α the inclination, t will be the horizontal trace of the line B A produced, and $p\,t$ the *length of plan* of the line B t, and similarly s and $q\,s$ with respect to the line B E produced. The horizontal trace of the plane containing B A, B E must pass through t and s (Theor. 24, *p*. 26). N o is this horizontal trace, and a ground line ($x\,y$) is assumed anywhere at right angles to it. Now, since $t\,p$, $s\,q$ are the lengths of the plans of B t, B s, and t and s their traces; arcs of radii $t\,p$, $s\,q$, with centres t and s must intersect in b the *plan* of B, and if from b a perpendicular is drawn to $x\,y$, and the produced portion $\pi\,b'$ made equal to B p (the assumed height of B), then b' is the elevation of B (Theors. 2 and 4, *pp*. 16, 17), and $b'\,o$ is the vertical trace of the plane of the pentagon (note to Prob. XIV.). The points A ... E are now turned up into the plane $b'\,o$ N and their plans $a \ldots e$ deduced, as in Prob. XIV. Fig. 25, Pl. V.

A second elevation of the pentagon on an assumed ground line ($y\,z$) is added.

If θ be the angle contained by the two lines whose inclinations are given (A B E, in Fig. 25), then the problem is impossible, if $\alpha + \beta + \theta$ exceed 180°. Note.

The curve in which a curved surface is cut by a plane of projection is termed the *trace of the surface* on that plane. Definition. "Trace of a surface."

A *conical surface* is generated by a straight line passing through a fixed point, and subject to any other condition whatever. Definition. "Conical surface."

Definition. "Cylindrical surface." A *cylindrical surface* is generated by a straight line moving parallel to a fixed straight line, and subject to any other condition.

Theorem 31. If a plane touches a conical or cylindrical surface, its trace on any plane must be tangent to the trace of the surface on the same plane.

Problem XVI. Determine the plane or planes inclined at a given angle and making a given angle with a given plane.

Fig. 27, Pl. VI. M O N (Fig. 27, Pl. VI.) is the given plane assumed at right angles to the vertical plane of projection. Then if any point *v′ v* (assumed external to this plane) be made the vertex of two right cones, one of which has its base on the horizontal plane and the other on the given plane M O N, the generatrix of the former making the given angle β with the horizontal plane, and of the latter the other given angle a with the plane M O N; then all planes touching these two right conical surfaces will fulfil the required conditions (Theor. 29, *p.* 27), and their horizontal traces must be tangent to the horizontal traces of the surfaces (Theor. 31). The trace of the first conical surface is the circle of its base, and the trace of the second surface is the section of that surface by the horizontal plane of projection, and is in this case an ellipse. The line *v′ t* drawn from *v′* at an angle β to the ground line determines *t f* the radius of the base of the first cone: this base can then be described with centre *v*. To determine the elliptic trace of the second conical surface, the lines *v′ m*, *v′ n* are drawn from *v′*, making angles of a with O M, and produced to meet the ground line in *a′*, *b′*, then *a′ v′ b′* is the elevation of that surface and *v′ q* produced is the elevation of its axis. Now a vertical plane containing the axes (*v′ f* and *v′ q*) of the two conical surfaces would cut the second surface in lines whose elevations are *v′ m a′*, *v′ n b′*. The same plane would cut the horizontal plane in the line *z z*. The horizontal traces *a* and *b* of the lines *v′ m a′*, *v′ n b′* are the extremities of the major axis of the elliptic trace, and if *a b* is bisected by a perpendicular $o\,o'$, then $v'\,o'$ and $o'\,o$ will be the traces of a plane containing the minor axis. This plane cuts the conical surface in two lines, of which $v'\,o'$ is the common elevation, and whose horizontal traces are the extremities of the minor axis of the ellipse. To determine the length of the minor axis, produce $v'\,q$ and through o' draw a line perpendicular to $v'\,q$ produced, cutting it in r and $v'\,b'$ in s. Then $r\,s$ is the radius of the circle which is the section of the cone $a'\,v'\,b'$ by a plane passing through o' and perpendicular to the axis $v'\,r$. Now the plane $v'\,o'\,o$ cuts this circular section in a chord of which o' is the elevation, and this chord must be equal to the minor axis. With centre r and radius $r\,s$ describe an arc of a circle, and from o' draw a line at right angles to $r\,s$, cutting this arc in p. Then the arc represents a portion of the circular section of the cone $a'\,v'\,b'$ rotated into a vertical plane about $r\,s$. Hence $o'\,p$ is half the length of the required chord. Make $o\,d_1$, $o\,d_2$ equal to $o'\,p$, and complete the ellipse $a\,d_1\,b\,d_2$ with $a\,b$, $d_1\,d_2$ as axes. All lines which can be drawn touching this ellipse and the circle $h\,h$ will be the horizontal traces of planes fulfilling the required conditions (Theor. 31, *p.* 38). The vertical traces of these planes may be determined from the condition that the planes must all contain the point V. Two planes only are shewn in the figure, though four are possible.

If the circle and ellipse touch each other externally, there are three planes; if internally, there is one plane. If the circle and ellipse cut each other there are two planes. If one falls wholly within the other the problem is impossible. Note *a.*

The trace of the second conical surface may, by varying the data, be a parabola or a hyperbola, but the principles explained are equally applicable, since either curve can be drawn if its axis ($z\,z$), its vertex (a), and a chord ($d_1\,d_2$) are known. Note *b.*

THE PROJECTION OF CIRCLES.

Theorem 32. The projection of a circle is either *an equal circle*, *a line equal to the diameter*, or *an ellipse*, according as its plane is *parallel*, *perpendicular*, or *oblique* to the plane of projection. In the latter case the major axis of the ellipse is always equal to the diameter of the circle, and the minor axis to that diameter multiplied by the cosine of the angle of inclination.

Thus the projection of a circle on any plane can be determined if the inclination of its plane to the plane of projection is known; and conversely, if the major and minor axes of the ellipse which is the projection of a circle on any plane, are known, the inclination of the plane of that circle to the plane on which it is projected can be obtained.

Note. It should be noticed that the major axis of the elliptic projection of any circle is the projection of that diameter of the circle which is parallel to the plane of projection. Further, the projections of any two diameters of a circle which are mutually at right angles will be a pair of conjugate diameters of the elliptic projection of that circle. It will frequently be best to determine the elliptic projection of a circle from a pair of conjugate diameters thus obtained. In Fig. 26, Pl. V., the line $a' b'$ is the elevation of a circle whose plane is inclined α to the horizontal plane. Hence $c\ c$ equal to $a' b'$ and set off on the line projected from c' the elevation of the centre of the circle, is the major axis of the elliptic plan of the latter, and $a\ b$ bisecting $c\ c$ at right angles is its minor axis. To draw a new elevation on a plane $z\ z$ it is merely necessary to obtain $c'' c''$, $a'' b''$, the elevations of $c\ c$, $a\ b$, and to draw the ellipse of which these lines are conjugate diameters.

Fig. 26, Pl. V.

Thus the elevation on $z\ z$ is drawn without determining the inclination of the plane of the circle to the plane $z\ z$. A third elevation of the same circle on $l\ l$, similarly obtained, is shown.

EXAMPLES.

1. The two traces of a plane each make 40° with $x\ y$; from any external point draw a line inclined at 25° and parallel to the plane. Determine also a second plane containing the line and at right angles to the first plane. (Prob. IX. From any point on the line drop a perpendicular to the plane, Theor. 28, *p.* 26, then determine a plane containing the line and the perpendicular.)

2. Assume the projections of three points, and determine the plane containing them. (Prob. X.)

3. From a point, 2″ from the vertical and 3″ above the horizontal plane of projection, draw a line making 60° with a plane whose traces make 45° with $x\ y$. Shew the intersection of the line and plane. (Converse of Prob. VII. An infinite number of lines fulfil the conditions.)

4. Three lines mutually at right angles meet in a point 2·5″ from both planes of projection. Two of them are inclined at 35° and 40° respectively. Determine the plans of all three lines and also their elevations on a plane making 20° with one of them. (Determine the two lines whose inclinations are given by Prob. XV., draw the third line perpendicular to the plane containing the two first, Theor. 28.)

5. Determine planes inclined at 78° and making 65° with a plane inclined at 40°. (Prob. XVI.)

6. Two planes at right angles to each other intersect in a line inclined at 35°. One of these planes is inclined at 55°. Determine the traces of both. (Converse of Prob. V.)

7. Assume the projections of a line not parallel to either plane of projection. From any external point drop a perpendicular on this line. (Determine a plane containing the point and perpendicular to the line, Theor. 28, *p.* 26. Find the intersection of the plane and line, Prob. VI.; join this intersection and the assumed external point.)

8. Three planes are mutually at right angles. Two of them are inclined at 30° and 75° respectively. All meet in a point 2·5″ from both planes of projection. Determine the traces of all three. (First, determine one of the planes whose inclinations are given. Then through the given point draw a line perpendicular to this plane, Theor. 28, *p.* 26. Determine a plane containing this perpendicular and inclined at the other given angle. Lastly, determine the intersection of these two planes, and draw a third plane through the given point and at right angles to this intersection.)

9. Determine the traces of a plane inclined 70° and 55° to the horizontal and vertical planes of projection respectively. Determine also the real angle contained by these traces. (Prob. IV.)

10. Assume a plane inclined at 60°, place in it a line inclined at 30°, and determine a plane containing this line and perpendicular to the first plane. (Prob. VIII. Then from any point on the line draw a perpendicular to the plane, and determine a second plane containing the line and this perpendicular, Prob. X.)

11. The horizontal and vertical traces of a plane make 40° and 60° with $x\,y$ respectively. Determine the inclination of this plane to the planes of projection, and determine a second plane parallel to and 2″ from it. (Prob. III. Then at any point in the plane raise a perpendicular 2″ long, and determine a second plane passing through the extremity of this perpendicular and parallel to the first plane, Theor. 25, *p.* 26.)

12. Assume the projections of a line not parallel to either plane of projection. Determine the traces of a plane perpendicular to this line and cutting it in a point 2″ above the horizontal plane.

13. Four points whose heights are 2″, 0·5″, 3·5″, and 1·2″, have their plans at the angles of an irregular four-sided figure $a\,b\,c\,d$; $a\,b = b\,c = 2{\cdot}5''$; $c\,d = d\,a = 3''$; $b\,d = 4''$. Determine the sphere containing the four points. (Find the elevations of the points and join them.

Determine three planes at right angles to and bisecting any three of the lines thus obtained. The intersection of these three planes is the centre of the sphere, and the distance from this intersection to any one of the given points is its radius.)

14. A plane has its horizontal and vertical traces inclined at 50° and 60° to $x y$ respectively. Determine a point in it, 2″ from the vertical and 2·5″ from the horizontal plane of projection. From this point draw two lines in the plane inclined at 25°. Determine a second plane bisecting the angle contained by the lines and perpendicular to the first plane. (Prob. VIII. Draw the line bisecting the angle contained by the two lines, and determine the plane containing this line and perpendicular to the given plane.)

15. Assume a plane whose traces make 45° with $x y$, and from any two external points draw lines meeting in a point on this plane and equally inclined to it. (From one of the assumed points drop a perpendicular on the given plane, find its intersection with the plane, Prob. VI. Then produce the perpendicular, making the produced part equal to the perpendicular. Join the extremity of the line so obtained with the other external point, and find the intersection of the joining line with the given plane.)

16. Draw a line inclined 30° and 40° to the vertical and horizontal planes respectively. Draw a second line parallel to both planes of projection and 3″ from each. Determine the shortest distance between the two lines. (This shortest distance is the line perpendicular to both lines.)

17. From two given points draw two lines which meet on a given plane and make equal angles with that plane. (From either point drop a perpendicular on the plane and find its intersection with the plane (Prob. VI.). Produce the perpendicular, making the produced portion equal to the perpendicular. Join the extremity of the produced portion to the second point, and determine the intersection of the line so obtained with the given plane.)

18. An equilateral triangle of 3″ side has its angles at 0·5″, 1·5″, and 2·5″ respectively, above the horizontal plane. Draw its plan and also an elevation on a plane inclined 35° to the plan of one of its sides.

19. Draw the plan of a pentagon of 1·5″ side, its plane being inclined at 45° and one side at 20°. Draw also elevations on planes parallel to any two sides. (Prob. XIV.)

20. Draw the plan and elevations of the same pentagon when one diagonal is inclined at 20°. (Prob. XIV.)

21. A circle 3″ diameter has its plane inclined 60°. Draw its plan and also an elevation on a plane inclined 50° to its horizontal diameter. (Note on Projection of Circles.)

22. Two diameters of a circle are inclined at 40° and 25°; these diameters make 65° with each other. The radius of the circle is 1·25″. Draw its plan and also an elevation on a plane parallel to one of these

diameters. (Determine the plane containing the diameters, Prob. XV. Through the intersection of the diameters draw a horizontal line in this plane 2·5″ long; this is the major axis of the plan of the circle. The inclination of the plane containing the circle determines the minor axis.)

23. A heptagon (side 1″) has two adjacent sides inclined at 15° and 25°. Draw its plan and also an elevation on a plane inclined at 40° to the plan of any diagonal. (Prob. XV.)

24. Draw the plan of the same heptagon when one side and an adjacent diagonal are inclined at 30° and 40° respectively.

CHAPTER IV.

THE USE OF INDICES.

POINTS, lines and planes in space may be correctly represented on paper by a method differing somewhat from that explained in the previous chapters. This method, which is sometimes termed "horizontal projection," possesses considerable advantages in simplifying constructions, and has moreover several directly practical applications. A horizontal plane of projection only is required, the regular vertical plane being dispensed with, although elevations on vertical planes will still be occasionally necessary. As in the preceding chapters, points, lines and planes are treated in turn.

Points.

If to the plan of any point a number is affixed, denoting in terms of any known unit the height of that point above the horizontal plane of projection, then the point is fully determined and its position in space is known. The number affixed is termed the ***index of the point;*** and in order to distinguish between points *above* and *below* the horizontal plane, the sign – is prefixed to the indices of the latter. It will be seen, therefore, that the index performs the function of the vertical plane of projection in giving the heights of points.

Index of a point.

Fig. 28, Pl. VII.

In Fig. 28, Pl. VII., a series of points (the same as in Fig. 1, Pl. I.) are shewn by their plans and elevations, and also by their figured plans only, the unit assumed for their indices being 0·1 inch.

Lines.

Fig. 29, Pl. VII.

A line is fully determined if its plan and the indices of any two of its points are known. Thus in Fig. 29, Pl. VII., a line A B is shewn by its projections *a b*, *a′ b′*, and also by its figured plan *a* 7, *b* –5.

The horizontal trace of a line will be that point whose index is 0. **Note *a.***

All points on a horizontal line have the same index. **Note *b.***

If m is the distance between the figured plans of any two points on a line, and n the *difference* of their indices; then $\frac{n}{m}$ = tangent of inclination of line. Thus, to determine the inclination of a line, it is merely necessary to construct a right-angled triangle having a base equal to the plan of any segment of the line and height equal to the difference of the indices of the ends of that segment. The angle subtended by the height is the inclination of the line. (See Prob. II., Chap. II.) **Note *c.***

Planes. If the plans and indices of any two horizontal lines lying in a plane are known, the plane is fully determined in position and inclination. Thus, in Fig. 31, Pl. VII., *a b* and *c d*, two horizontals whose indices are 10 and 20, determine a plane the tangent of whose inclination is the difference of level of the two horizontals *a b*, *b c* (10 units), divided by their distance apart on plan (*b d*). Since horizontals with equidifferent indices must have equidistant plans, a line parallel to *a b*, and at a distance from it equal to *b d*, will be the horizontal whose index is 0, and similarly the horizontals whose indices are – 10, 30, 40, &c., can be determined. **Fig. 31, Pl. VII.**

The horizontal whose index is 0 is the *horizontal trace* of the plane. **Note *d.***

"Scale of slope." Instead of drawing the horizontals of a plane, it is more convenient to use what is termed a *scale of slope*, constructed on a line anywhere at right angles to the horizontals. The points at which horizontals at equidifferent levels cut this line, give the divisions of the scale, each of which divisions is figured with the index of the horizontal passing through it. To distinguish the scale of slope from the mere figured plan of a line a second line is added; and in order that the *direction* of the slope of a plane may be immediately apparent from its scale of slope, one of the lines of the latter is strengthened, the stronger line being always on the *left* side in the ascending direction of the plane. In Fig. 31 the scale of slope of the plane, of which the lines ab, cd are horizontals, is shewn.

Note *e.* The inclination of a plane can be at once determined from its scale of slope since it is merely necessary to construct a right-angled triangle whose base is equal to the distance apart of any two horizontals, and whose height is equal to the difference of their indices. Thus, in Fig. 31, making ss' at right angles to the scale of slope equal to $30 - 10$ units, and joining $s't$, we obtain θ the inclination of the plane. In other words, the scale of slope is considered as a line and its inclination obtained.

Note *f.* Lines perpendicular to a plane have *plans* parallel to its scale of slope. This follows from Theorem 28, p. 26.

In the following problems the unit assumed for the indices is 0·1 inch; thus a point whose index is 28 will be 2·8 inches above the horizontal plane of projection.

Problem I. **Given the plan of a line and the indices of any two points, to determine the index of any third point on the line.**

Fig. 30, Pl. VII. ab, Fig. 30, Pl. VII., is the given plan (the indices of a and b being 5 and 19 respectively), x the point whose index is to be determined. Using ab as a ground line of level 5, draw bb' at right angles to ab, and in length equal to 14 units—the difference of the indices of a and b. Then b' is the elevation of b, and $b'a$ is the elevation of the line ab; while the length of the perpendicular xx', read off from the scale of units, is the *difference* of the indices of a and x. The real index of x is therefore $xx' + 5$.

Since by similar triangles $\frac{xx'}{ax} = \frac{bb'}{ab}$, the *difference* of the indices of x and a is a fourth proportional to the horizontal distance between a and b, the difference of the indices of a and b, and the distance ax; and generally the difference of the indices of any two points on a line is proportional to the distance apart of their plans.

Thus the index of any point on a line may be determined without construction by means of the sector. The converse of the above process, determining the plans (y, z) corresponding to any assigned indices (0, 11·5), needs no explanation.

The above method applied to one of the scale of slope lines of a plane serves to determine the index of any assumed horizontal, or the horizontal with any assumed index. **Note a.**

The angle $b' a b$ (a in Fig. 30) is the inclination of the line $a b$. **Note b.**

The following problems should be carefully compared with the corresponding problems of Chaps. II. and III.

It will be seen that, although the vertical co-ordinate plane is dispensed with, it is frequently necessary to resort to elevations made on the plans of lines or scales of slope of planes, used as ground lines. In these cases it is sometimes advisable to assume the ground line to have some particular level. This merely amounts to raising or lowering the horizontal plane and does not in any way affect results, provided that it is always remembered that in determining the indices of points from elevations made on such a ground line, these indices will be the *heights of the elevations above the ground line + the index of that ground line.* Thus, in Fig. 30, the index of the point x is not $x x'$, but $x x' + 5$.

Draw a line through a given point parallel to a given line. **Problem II.**

Fig. 29, Pl. VII.

In Fig. 29, Pl. VII., ab is the given line, p 35 the given point. The plan of the required line must of course be parallel to $a b$; and as the *inclinations* of parallel lines are the same, the *horizontal distance* (that is the distance on plan) corresponding to any difference of level is the same on both lines. Thus making $p q$ equal to $a b$, we obtain a point q, whose index will be $35 - 12 = 23$.

Determine the angle contained between two given lines. **Problem III.**

Fig. 32, Pl. VIII.

$a b, c d$ (Fig. 32, Pl. VIII.) are the two lines intersecting in i 17. Determine two points (a x) having any the same index (5), one on each line (Prob. I.), and turn down the triangle a I x into the horizontal plane containing a and x. Draw $i s$ perpendicular to $a x$ and produce it; make $i i'$ at right angles to $i s$ equal to 12 units (the difference of the indices of i and s), thus obtaining $i's$ the real length of is. Make $s\,\mathrm{I}$ equal to $i's$ and join $\mathrm{I}a$, $\mathrm{I}\alpha$.

The angle $a\,\mathrm{I}\,\alpha\,(\theta)$ is then the angle contained by the lines ab, cd. (Compare Prob. V., Chap. II.)

Problem IV. **Determine the plane containing three given points.**

Fig. 33, Pl. VIII. By joining the points three lines are obtained, and if on any two of these lines points with the same index are determined, the line joining them is a horizontal of the required plane. A parallel line from any one of the given points is a second horizontal, and the scale of slope can therefore be drawn. (Compare Prob. X., Chap. III.)

Problem V. **Determine the intersection of a line and plane.**

Fig. 34, Pl. VIII. In Fig. 34, Pl. VIII., ab is the given line, S the scale of slope of the given plane. The fine line of the scale of slope is taken as a ground line of level 5, and on this line elevations of the given line and plane are made. Their intersection i' is the elevation, and therefore i the plan of the intersection of the line and plane. (Compare Prob. VI., Chap. III.)

Note *a*. The index of a being -2, its elevation must be taken *below* the ground line, that is to say, on the opposite side of the ground line to the elevation of b.

Note *b*. The index of i is the length $i'\pi + 5 = 11$ units.

Problem VI. **Determine the intersection of two planes and the angle they contain.**

Fig. 35. Pl. VIII. M and N (Fig. 35, Pl. VIII.) are the scales of slope of the given planes. The intersection of any two horizontals of the same index determines a point in the intersection of the planes. Thus a and b (the intersections of horizontals at levels 5 and 20 respectively) determine the line ab, which is the required intersection of planes M and N. The remaining construction corresponds exactly to that of Prob. V., Chap. II. Make an elevation (ab') of $a\,b$ on ab used as a ground line of level 5. Draw any line sr at right angles to ab and cutting ab in p and

the horizontals 5 of the two given planes in *s* and *r* respectively. Then *s r* is a horizontal (at level 5) of some plane at right angles to *a b*, and therefore at right angles to both the planes M and N; while *i'*, determined by a perpendicular from *p* on *a b'*, is the elevation of the intersection of this perpendicular plane with *a b*. Then *p* I, equal to *p i'*, and set off from *p* along *a b*, determines the turned-down triangle *s* I *r*, and the angle *s* I *r* (θ) is the required angle contained by the two planes. (Compare Prob. V. Chap. III.)

Note. The method of determining the intersection of planes from their horizontals has several useful applications. Thus the plans of the ridges, hips and valleys of a complicated system of roofs may be readily obtained by drawing a couple of horizontals on each slope.

Problem VII. **Determine the angle made by a given line with a given plane.**

Fig. 36, Pl. VIII. In Fig. 36, Pl. VIII., *a b* is the given line and M the scale of slope of the given plane. Then the angle required is that made by the line *a b*, with a second line joining the point in which *a b* cuts the plane M and the foot of the perpendicular dropped from any point of *a b* on the plane M (Euc. XI., def. 5).

Elevations of *a b* and the plane M are made on one of the scale of slope lines used as a ground line. The intersection of these elevations determines *i'* the elevation of the intersection of the line and plane, and *i* its plan is found by projection from *i'*. From *a*, a perpendicular is dropped on the plane; of this perpendicular *a' p'* is the elevation, and *a p, parallel to the scale of slope,* the plan. The right-angled triangle A P I may then be solved by determining the real length of any two of its sides, and the angle A I P (θ) is the angle required. (Compare Prob. VII., Chap. III.)

Note. In commencing any of the varied problems relating to lines and planes, and indeed any problem of solid geometry whatever, the student is again most strongly advised to start with a clear conception of the method he intends to pursue. By extemporizing inclined planes of paper and using pencils, pieces of wire, or thread, to represent lines as before suggested, it will generally be easy to arrive at a clear notion of the best course to adopt in any particular case.

EXAMPLES.

(Unit 0·1″.)

1. A line is inclined at 55°, draw its plan and figure on it points whose indices are – 7, 0, 13, 25. Draw the scale of slope of a plane cutting the line in the point 13 and perpendicular to it. (Make an elevation of the line on its plan as ground line. Through the elevation of point 13 draw a line perpendicular to the elevation of the line. This perpendicular will be the vertical trace of the required plane on the vertical plane containing the line. From this trace any number of horizontals of the plane can be determined.)

2. The plans of two intersecting lines make an angle of 80°; the lines are inclined at 30° and 50° respectively; the index of their intersection is 25. Determine the angle they contain (Prob. III.).

3. A line is inclined at 40°. Determine a second line inclined at 30°, and making 70° with the first line. (Converse of Prob. III.)

4. The plans of three points are situated at the angles of a triangle whose sides are 2″, 3″, and 3¼″; the indices of the points are 4, 25, —8. Determine the plane containing the given points (Prob. IV.).

5. Assume a plane inclined at 60°. Draw a line in it inclined 28°, from a point whose index is 15. (Compare Prob. VIII., Chap. III.)

6. Determine two planes perpendicular to each other and inclined at 30° and 80°. Shew their intersection, and determine a third plane perpendicular to both, and cutting this intersection in a point whose index is 13.

7. Assume a plane inclined at 50°. Draw a horizontal line of level 10; the plan of this line makes 30° with the horizontals of the plane. Find the intersection of this horizontal line with the plane and the angle it makes with the plane (Prob. VII.).

8. The horizontals of two planes make angles of 70° with each other on plan. The planes are inclined at 40° and 55°. Find the intersection of the planes, the inclination of this intersection, and the angle the planes contain (Prob. VI.).

9. Three points have plans 2·5″, 3·5″, and 4″ apart. From any one of them draw a line perpendicular to the line joining the other two. (Determine a plane containing the point selected and perpendicular to the line joining the other two. Find the intersection of the plane and line, and join this intersection and the selected point.)

10. Assume two lines which are not parallel and do not meet. Determine two parallel planes each of which contains one of the assumed lines. (From any point on one of the lines (A B) draw a line parallel to the other line (C D). Determine a plane containing A B and the

parallel, and a second plane containing CD and parallel to the first plane.)

11. The plans of four points are situated at the angles of a rhombus of 3¼" side and 3¾" diagonal; the indices of the points are 30, 23, 10, 17. Determine a sphere to contain all four points. (Determine three planes at right angles to and bisecting the lines joining three pairs of points. The intersection of these three planes is the centre of the sphere.)

CHAPTER V.

THE PROJECTION OF SOLIDS.

THE use of the term "solid" in geometry is somewhat peculiar. Since geometry has nothing to do with solidity in the ordinary sense, the term must be considered to mean merely "surface enclosing space." In descriptive geometry, however, solids are usually assumed to possess opacity, and those edges which would be invisible are drawn with dotted lines. This will be found to aid the eye in recognising their forms.

Regular solids.

There are only five "regular solids," that is to say, solids contained by equal regular polygons and capable of being inscribed in a sphere.

1. The "tetrahedron," contained by four equal equilateral triangles.
2. The "cube," contained by six equal squares.
3. The "octohedron," contained by eight equal equilateral triangles.
4. The "dodecahedron," contained by twelve equal pentagons.
5. The "icosahedron," contained by twenty equal equilateral triangles.

The following are also contained by plane figures, viz.:

1. "Pyramid," right and oblique.
2. "Prism," right and oblique.

Surfaces of revolution.

"Solids" or "surfaces of revolution" are generated by the revolution of lines, curves, or plane figures about a fixed line termed the axis. They have this peculiarity, that all sections by planes perpendicular to the axis are circles.

The following are capable of being thus generated, viz.:—

"Sphere," "right cone," "right cylinder," "annulus," "spheroid."

Ruled surfaces. All surfaces generated by the motion of straight lines are termed "ruled surfaces." Of these surfaces some are capable of development and some are not.

Conical and Cylindrical surfaces. Among the first class are "conical surfaces," generated by a line passing through a fixed point, and "cylindrical surfaces," by a line parallel to a fixed line (or director), the generating line in each case being subject to any other condition whatever. The generating line being considered infinite, conical surfaces will always consist of two distinct portions or "sheets" having a common vertex in the fixed point.

Among ruled surfaces incapable of development are—

Helicoidal surfaces. 1. "Helicoidal surfaces," generated by lines passing through and making a constant angle with a fixed line, while moving uniformly around and along it.

Conoidal surfaces. 2. "Conoidal surfaces," generated by lines moving parallel to a fixed plane passing through a fixed axis, and subject to any other condition. Thus the "hyperbolic paraboloid" is generated by a line moving parallel to a fixed plane and meeting two fixed straight lines which are not parallel and do not meet.

Note. Several of the surfaces above mentioned are capable of generation in more than one way. Thus a sphere is generated by the revolution of a semicircle about a fixed diameter and also by the motion, parallel to itself, of a circle of varying radius.

It is now proposed to shew how the projections of the simple solids in various positions may be obtained. The data fixing the position of such solids will usually be referable to one of the five following cases, which will be considered in turn. In dealing with these solids, wooden models, or better still wire skeletons, will be found a great assistance.

One face in or parallel to the horizontal plane. **Case I.**

One edge in or parallel to the horizontal plane, the inclination of one of the faces containing this edge being given. **Case II.**

Case III. *Given the inclination of one face and of a line in its plane.*

Case IV. *Given the inclination of two edges or diagonals.*

Case V. *Given the inclinations of two faces.*

Note 1. [illegible]

Note 2. [illegible]

Problem I Case I **Draw the projections of 1st, a Right Prism; 2nd, a Dodecahedron; 3rd, an Icosahedron, with one face resting on the horizontal plane.**

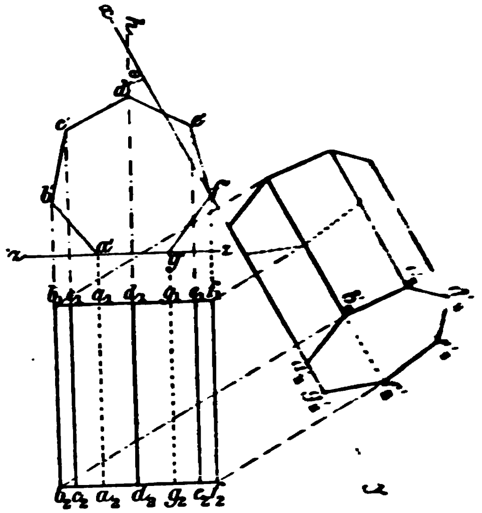
Right Prism. Fig. 87

1st. Suppose the prism to be hexagonal. By the conditions of the problem all the long edges will be horizontal, and their plans therefore equal to their real length, that is to the length of the prism. Moreover, the end of the prism will be vertical, and its elevation on a plane perpendicular to the axis will be a hexagon, which can be drawn at once.

Assume any ground line *xy* Fig. 87, and construct on it the hexagon *a* . . . *f*. This hexagon is the end elevation of the prism. From each of the points *a*, *b* . . . project down at right angles to *xy* and make $a_1 a_2$, $b_1 b_2$ . . . each equal to the given length of the prism. In other words, cut the projectors by two lines $b_1 f_1$, $b_2 f_2$ parallel to *xy* and at a distance apart equal to the length of the prism. The figure is now complete.

If any other condition were added—as for example that the axis of the prism should make an angle θ with xy—draw first the plan of the axis $h\,d_2$ making θ with xy. Draw zz anywhere at right angles to $h\,d_2$ and com-

Fig. 37.



plete the elevation and plan as before: then by the ordinary rules of projection obtain a new elevation on xy. In this case the elevation on zz may be regarded as auxiliary, being required in drawing the plan and determining the heights of all the points of the prism.

The two following solids will be best understood from wire models.

2nd. The method of obtaining the projections of a dodecahedron with one face on the horizontal plane is as follows:—

Draw two equal pentagons having the edge of the solid as side, inscribed in the same circle with angles alternating [illegible] apart; these pentagons will be the plans of the upper and lower horizontal faces of the solid. The sides of these pentagons will be sides of the ten other equal pentagons which complete the solid, and since each of these ten pentagons has one horizontal side, they will have also a horizontal diagonal, the plan of which will be equal to its true length. Hence surrounding *each* of the two pentagons first drawn by a similarly situated pentagon whose side is equal to a diagonal of the

Case III. *Given the inclination of one face and of a line in its plane.*

Case IV. *Given the inclination of two edges, or diagonals.*

Case V. *Given the inclinations of two faces.*

Note *a.* It is impossible to lay down any *absolute* rules to be followed in each of the above cases, as different solids will require slightly different treatment. It may be remarked, however, that whenever (as in Cases I. and II.) an edge or face is parallel to or lies in the horizontal plane, this edge or face having its *real* size in plan can be drawn at once. To complete the plan of the solid an elevation will generally be necessary, and for this purpose a vertical plane should be selected either parallel or perpendicular to the base or to one face, since an elevation on such a vertical plane can generally be drawn without difficulty. The above remarks will be made clear in the following examples.

Note *b.* In drawing the projections of a solid, the spectator is always supposed to be placed so as to look *through the solid to the plane of projection.* Thus the solid is supposed to be *between the spectator and the plane on which its projection is drawn.* Certain edges will therefore be invisible, and these edges are consequently shewn in dotted lines. If what has been said above with regard to the supposed position of the spectator is borne in mind, no difficulty will be found in distinguishing the invisible edges of any view of a solid in any position.

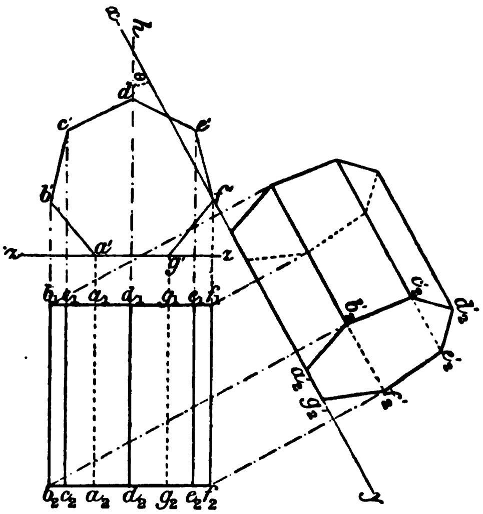
Problem I. Case I. **Draw the projections of, 1st, a Right Prism; 2nd, a Dodecahedron; 3rd, an Icosahedron, with one face resting on the horizontal plane.**

Right Prism. Fig. 37. 1st. Suppose the prism to be heptagonal. By the conditions of the problem all the long edges will be horizontal, and their plans therefore equal to their real length, that is to the height of the prism. Moreover, the end of the prism will be vertical, and its elevation on a plane perpendicular to the axis will be a heptagon, which can be drawn at once.

Assume any ground line $z z$ Fig. 37, and construct on it the heptagon $a' \ldots g'$. This heptagon is the end elevation of the prism. From each of the points a', b' . . . project down at right angles to $z z$ and make $a_1 a_2$, $b_1 b_2$ . . . each equal to the given height of the prism. In other words, cut the projectors by two lines $b_1 f_1$, $b_2 f_2$ parallel to $z z$, and at a distance apart equal to the height of the prism. The figure is now complete.

If any other condition were added—as for example that the axis of the prism should make an angle θ with xy—draw first the plan of the axis $h\,d_2$, making θ with xy. Draw $z\,z$ anywhere at right angles to $h\,d_2$, and complete the elevation and plan as before: then by the ordinary rules of projection obtain a new elevation on xy. In this case the elevation on $z\,z$ may be regarded as auxiliary, being required in drawing the plan and determining the heights of all the points of the prism.

Fig. 37.



The two following solids will be best understood from wire models.

2nd. The method of obtaining the projections of a dodecahedron with one face on the horizontal plane is as follows :—

Projections of a dodecahedron. Fig. 47, Pl. XII.

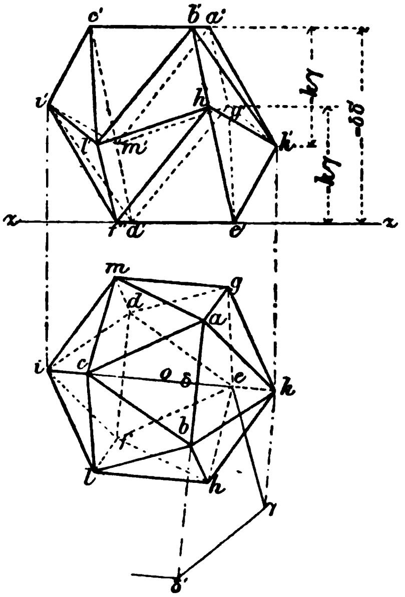
Draw two equal pentagons having the edge of the solid as side, inscribed in the same circle with angles alternating and at equal distances apart; these pentagons will be the *plans* of the upper and lower horizontal faces of the solid. The sides of these pentagons will be sides of the ten other equal pentagons which complete the solid, and since each of these ten pentagons has one horizontal side, they will have also a horizontal diagonal, the plan of which will be equal to its real length. Hence surrounding *each* of the two pentagons first drawn by a similarly situated pentagon whose side is equal to a diagonal of the small pentagons, we obtain the remaining ten angular points of the solid. To complete the plan of the solid it is merely necessary to join the angles of the large pentagons to those of their respective small pentagons, as shewn in Fig. 47, Pl. XII. To determine the elevation it is necessary to obtain the relative heights of the planes of the four pentagons, and it is evident from the symmetry of the figure that the distances of the planes of the upper small pentagon from the upper large pentagon, and of the lower small pentagon from the lower large pentagon, are equal. From *a* (Fig. 47, Pl. XII.), one of the angles of the lower large pentagon drop a perpendicular *a b* on the side of the upper small pentagon. (This perpendicular coincides with *a c* the plan of one of the *edges* of the solid.) Then since the real lengths of *a c* and *a b* are known, the *vertical distances* of *b* above *a* and *a* above *c* can be determined. The construction is shewn, and we thus obtain *b b′* as the whole height of the solid, and *a a′* as the height of the lower large pentagon. Draw two lines parallel to *x y* and at distances above it equal to *b′ b*, *a′ a*, and a third line at a distance below the upper of the first two lines equal to *a′ a*. Then the elevations of all the points of the solid are obtained by projecting from their plans on to these three lines and *x y*, taking care of course to project from the points of the lower large pentagon on to the first line above *x y*, and so on. (In the figure these three lines and the sides of the large pentagons are not shewn.)

Icosahedron. 3rd. As in the case of the dodecahedron, the plan of the icosahedron can be drawn at once from the symmetry of the solid without having recourse to an auxiliary elevation.

Fig. 38. Referring to Fig. 38, or to a model, it will be seen that the solid has twelve angular points lying in four different planes, three in each. Moreover, when it is resting on one face (as in Fig. 38), the angles of the upper and lower equilateral triangles alternate, forming a hexagon on plan, while the plans of the remaining six angles of the solid lie on a larger concentric hexagon whose sides are parallel to those of the first. Commence therefore by drawing the two equilateral triangles *a b c*, *d e f*, with sides equal to the given edge of the icosahedron, and place them so that *a e b f c d* is a hexagon. From the centre *o* of the hexagon draw six radiating lines passing through the points *a . . . f*. These lines will give the directions of the plans of six edges of the solid. Now the points *a b h e g* (Fig. 38) lie on a pentagon with the edge of the solid as side, and the diagonal *g h* of this pentagon is horizontal, and consequently its plan will be equal to its real length. Obtain by construction the length of the diagonal of a pentagon having a side equal to *a b*, and place this length in between the radial lines *a g*, *b h*, making it parallel to *a b*, as was done in the case of the dodecahedron. This fixes the positions of the points *g* and *h*. With *o* as centre and *o g* (or *o h*) as radius, describe a circle cutting the radial lines in *k*, *l*, *i*, *m*. All the points of the icosahedron are now determined, and it is required only to join *g k*, *a k*, *b k*, *k h*, and so on all round the figure, in order to

complete the plan. To draw the elevation on any ground line zz, it is necessary to obtain the heights of the three groups of points, $a\,b\,c$, $g\,h\,i$, $k\,l\,m$. From k draw a line perpendicular to $o\,k$, and cut it by an arc from centre e with radius equal to $a\,b$. Then $k\gamma$ is equal to the height of k, and consequently of l and m. Produce $a\,b$ and cut it, by an arc described with γ as centre and radius equal to $c\,\delta$ the perpendicular from c on $a\,b$. Then $\delta\,\delta'$ is equal to the height of the points a, b, and c. By the symmetry of the solid, the height of k, l, m, above the horizontal plane is equal to the distance of $h\,i\,g$ below $a\,b\,c$. Draw three lines parallel to zz, the outer one being at a distance from it equal to $\delta\,\delta'$, the two others at a distance of $k\gamma$ from the first and from zz respectively. By projecting all the points of the plan on to these lines respectively, the elevation can be completed.

Fig. 38.



Draw the plan and elevation of a Pentagonal Right Pyramid lying with one triangular face on the horizontal plane, the plan of the axis making an angle a with the vertical plane.

Problem II. Case I.

The plan of the axis being taken at an angle a to the

Fig. 39, Pl. IX.

ground line, the isosceles triangle *a b v* on which the solid rests can be drawn, since this triangle will evidently be symmetrical about the plan of the axis. The side *a b* is perpendicular to and bisected by the plan of the axis, and is equal to the given side of base of the pyramid; *a v, b v* are equal to the slant edges, and *v* is the vertex of the pyramid. The length of the slant edges is obtained by constructing a pentagon on *a b* and making V V′ at right angles to V C, equal to the given height of the pyramid. Then V′ C is the length of the slant edges. Now if the pyramid be supposed rotated about the edge *a b* till it stands upright on the horizontal plane, its base would coincide with the pentagon *a b* C D E, and as the points E, D, C must evidently move in vertical planes at right angles to the axis of rotation (*a b*), their plans when the pyramid is lying on *a b v* must be somewhere on the lines drawn from E, D, C at right angles to *ab*. (The line thus drawn from D will of course coincide with the plan of the axis.) Now if a vertical plane parallel to the axis (and therefore perpendicular to the base) of the pyramid be assumed, an elevation of the solid on such a plane can be drawn at once; *z z* parallel to the plan of the axis is the ground line of this assumed plane, and the points *v, a, b*, being *in* the horizontal plane, will have *v′, a′, b′*, as their elevations (Theor. 6, *p.* 17), the two latter coinciding. The elevation of the pyramid is then completed by making *a′ d′* equal to D π, *v′ d′* equal to *a v* or *b v* (the slant edge), and *a′ e′* equal to E ϵ, then *v′ e′* is the common elevation of the two edges *v e, v c*. The required plans *e, d, c*, must lie in perpendiculars drawn from *e′, d′, c′* (their elevations) to the ground line *z z*, and are, therefore, determined. Joining them with *v* the plan of the vertex, we have the complete plan of the solid. The elevation on *x y* is readily deduced by projecting from *a, b, c*, &c., and setting off the heights of these points taken from their elevations on *z z* (Theor. 4, *p.* 17).

Note *a*. The principle involved in the use of the pentagon *a b* C D E should be thoroughly understood, as it is always applicable in determining the plan of a polygon of which one edge is parallel to, or lies in the horizontal plane.

Note *b*. Having obtained the plan and *one* elevation of a solid, any number of elevations on any assumed vertical planes can be drawn by the application of Theor. 4, *p*. 17.

Problem III. Case II.

A Heptagonal Right Prism rests on an edge of its base, the face containing that edge being inclined at an angle α. Draw the plan of the solid, and also an elevation on a vertical plane, inclined β to the plan of its axis.

Fig. 40, Pl. IX.

The plan of the axis being taken at an angle β to xy, ab the edge on which the prism rests is drawn at right angles to and bisected by it. On ab, or (to avoid confusing the figure) on AB parallel and equal to ab, the heptagon A ..... G is described. The lines from A ..... G at right angles to ab will contain the required plans $a \ldots\ldots g$, and will coincide with the plans of the seven parallel edges of the prism. Assume zz as the ground line of a vertical plane parallel to the axis of the prism, and therefore perpendicular to the plane of its base and to the edge ab. Then a' will be the elevation of ab, and $a'l'$ equal to the given height of the prism, and making α with zz, will be the elevation of the face containing ab. Similarly, $a'e'$, at right angles to $a'l'$, and equal to Eϵ, is the elevation of the base of the prism: $a'g'$ and $a'f'$ equal to Gγ, Fδ respectively, determine g' and f' the elevations of the points g, c and f, d. Lines from g', f', and e' parallel and equal to $a'l'$, meeting $l'm'$ parallel and equal to $a'e'$, complete the elevation of the solid.

The *plan* of the heptagonal base is determined by projecting from g', f', e', on to the lines from A ..... G before mentioned. Since the plans of the seven parallel edges are equal (Theor. 15, *p*. 19), if one of them em is determined by projection from m', its length set off from the points $a \ldots\ldots g$ along the plans of the edges gives the plan of the upper heptagonal end of the prism. It remains to draw an elevation of the solid on the ground line xy. This elevation is shewn on the figure; the construction for obtaining it needs no explanation.

Note. In the last two problems, the reasons for selecting the two vertical planes *z z* for the elevations are evident. The elevations on these two planes are the simplest possible, and can be obtained merely from the known dimensions of the solids. If it is required to determine the plan of a prism of which the axis is horizontal (the inclination of one of the *rectangular faces* being given), it would be necessary to assume a vertical plane at right angles to the axis. The elevation on such a plane would be a polygon equal to that of the base or end of the prism, and having one side inclined to the ground line at the given angle. The construction would in other respects be identical with Problem I.

Problem IV. Case III.

Determine the plan of a Tetrahedron, one face being inclined at an angle α, and one edge of that face at an angle β.

Fig. 41, Pl. X.

The construction for determining the plan *a b c* of the triangular face A B C is shewn, but as it is identical with that of Prob. XIV., Chap. III., it is not explained. It remains, therefore, to determine *d* the plan of D the fourth angle or vertex of the tetrahedron, and for this purpose it is necessary to obtain the height or perpendicular distance of the point D from the face A B C. A B C (Fig. 42) is the plan of the tetrahedron when standing on the vertical plane; D, the plan of its vertex or fourth angular point, evidently coinciding with the centre of the circumscribing circle of the triangle A B C. Now the *real length* of D C is that of an edge of the tetrahedron. Hence, constructing a right-angled triangle of which D C is the base and $d''c$ equal to A C (an edge of the solid) the hypothenuse, we have d'' D as the *height* of D above the plane of the triangle A B C. Returning to Fig. 41, O is the centre of the turned-down triangle A B C, and o' and *o* its elevation and plan when turned up into the plane L M N; we have to erect at o' *o* a perpendicular to the plane L M N equal to D d'' (Fig. 42). The projections of this perpendicular are at right angles to the traces of the plane (Theor. 28, *p.* 26); and since the perpendicular is parallel to the vertical plane of projection, its *real length* (D d'') can be set off from o' (Theor. 12, *p.* 18), thus determining d', and hence by projection *d*.

Fig. 42, Pl. X.

Fig. 41, Pl. X.

It only remains to draw $d\,a$, $d\,b$, $d\,c$ on plan, and $d'\,a'$, $d'\,b'$, $d'\,c'$ on elevation.

If the angle β is greater than the angle α the problem is impossible. **Note.**

Draw the plan of an Octohedron, given the inclinations, α, β, of two adjacent edges. **Problem V. Case IV.**

The construction for determining the plan of any plane polygon when the inclinations of two of its lines are given, is explained in Prob. XV., Chap. III.

In Fig. 43, Pl. X., $a\,b\,c$ is the plan and $a'\,b'\,c'$ the elevation of the face A B C, of which the edges A B and A C are inclined at the given angles α and β respectively. The construction for determining this face and the plane $l\,o\,m$ containing it, is shewn, but needs no explanation, as it is identical with that of Fig. 25, Pl. V. The remaining three angles D, E, F of the solid lie in a plane parallel to $l\,o\,m$. To determine this plane it is necessary to know the height of the solid, i. e. the perpendicular distance between its parallel faces. Fig. 44 is the plan of the octohedron when one face A B C is horizontal. This plan is readily drawn, since all its angular points lie on the circle circumscribing the triangle A B C and form the hexagon A....F. Join D C, cutting B A in q, and E F in p. The *real length* of Dq is equal to Dp. Hence, if a right-angled triangle be drawn on Dq as base, and having an hypothenuse $q\,d'$ equal to Dp, the side Dd' will be the vertical height of D above q, i. e. the perpendicular distance apart of the parallel faces A B C, D E F. Returning to Fig. 43, find G the centre of the circumscribing circle of the turned-down triangle A B C. Then turning G up into the plane $l\,o\,m$, we obtain $g_1\,g'_1$, its projections. From the point $g_1\,g'_1$, draw a perpendicular to the plane $l\,o\,m$, equal in length to D d' (Fig. 44), the distance apart of the parallel faces. The projections of this perpendicular are at right angles to the traces of the plane (Theor. 28, *p.* 26), and its elevation $g'_1\,g'_2$ is equal to its real length Dd' (Theor. 12, *p.* 18). Then the point $g_2\,g'_2$, the extremity of this perpendicular, is the centre of

Fig. 43, Pl. X.

Fig. 44, Pl. X.

Fig. 43, Pl. X.

the circumscribing circle of the face D E F of which the plan is required. Join g_1, b, then a line from g_2 parallel and equal to $g_1 b$, but drawn in the opposite direction, determines f one of the angles of the required face def. This face can now be completed by joining $g_1 a$ and $g_1 c$, and proceeding as before, or by drawing lines parallel and equal to ab, bc, ca, commencing at f. Join eb, ea, bd, cf, cd, af, thus completing the plan. The elevations d', e', f', can be obtained by projecting from the plans d, e, f, on to a line through g'_2 parallel to ol. (This line is of course the vertical trace of the plane containing the face D E F.)

Note *a*. The symmetry of a solid will often suggest several ways of completing its projections, but in all cases the plan of the face determined by the given conditions should be obtained first.

Note *b*. The problem is impossible if the angles B A C $+ \alpha + \beta$ exceed 180°.

Problem VI. Case V. **Determine the projections of a right regular hexagonal prism, given the inclinations α, β, of one face and the base respectively.**

Fig. 45, Pl. XI. In Fig. 45, Pl. XI., $h'ot$ is the plane of the face of the prism inclined α. A point vv' is assumed anywhere external to this plane, and is made the vertex of a right cone standing on the horizontal plane and having an angle at the base equal to β. The elevation of this cone is the isosceles triangle $v'm'n'$, and the circle mn described with centre v is its plan. Then the plane of the base must touch this cone (Theor. 29, *p.* 27). From the point $v'v$, a perpendicular is drawn to the plane $h'ot$; $v's'$, vs are the projections of this perpendicular (Theor. 28, *p.* 26) intersecting the plane $h'ot$ in aa'. Then since the plane of the base of the prism is at right angles to the plane of the face, it must contain the perpendicular $v's'$, vs. Hence the horizontal trace of the plane of the base must pass through p the horizontal trace of this perpendicular (Theor. 24, *p.* 26), and must touch the circle mn (Theor. 29, *p.* 27). Thus pt is the horizontal trace of the plane of the base, and the intersection of this plane with the plane $h'ot$ will be one edge of the base of the solid. The *plan* of this intersection must pass through *t* the intersection of the horizontal traces of the two planes (Prob. II., Chap. III.), and must contain *a* the plan of the intersection of the perpendicular *v′ s′*, *v s* with the plane *h′ o t*. Thus the line *a t* is the plan of the intersection of the two planes. Turning down the line whose plan is *a t* about *o t* the horizontal trace of the plane containing it, we obtain A *t*. Produce *t* A to B, making A B equal to a side of the base of the prism. On A B construct the rectangle A B G H, the sides A G, B H being made equal to the length of the prism; turn up the rectangle A B G H, and the parallelogram *a b g h* is obtained. This parallelogram is the plan of one of the faces of the prism and *a′ g′ b′ h′*, obtained by projecting on to the vertical trace of the plane containing this face, is its elevation. To complete the plan of the prism, turn down *a b* about *t p* the horizontal trace of the plane of the base, and on the turned-down line A′ B′ construct the hexagon A′ . . . . F′. This hexagon is the base of the prism turned down about the horizontal trace of the plane containing it, and therefore lines from the points A′ . . . . F′ at right angles to *t p* determine the plans of the long edges of the prism (Note *d*, *p*. 36). Making *a′ e′* (on the elevation of the perpendicular *a′ s′*) equal to B′ D′ the distance apart of the parallel sides of the hexagon and projecting, we obtain *e*, which must be one of the angles of the base of the prism, and *b d* equal and parallel to *a e* determines another point *d*. (The two points *e* and *d* must also lie on the perpendiculars previously drawn from E′ and D′ to the horizontal trace *t p*.) The remaining points (*c*, *f*) of the plan of the base must by the symmetry of the figure lie on the line bisecting *a e* and *b d*, and also on the perpendiculars from E′ and F′ respectively. The plan of the base is therefore complete, and it remains only to set off from each of the points *c*, *d*, *e*, *f*, distances equal to *a g* or *b h* along the perpendiculars from C′, D′, E′, F′. Joining the points so determined, the plan of the upper end of the prism is obtained.

To complete the elevation, lines from *a'*, *g'*, *b'*, *h'* parallel and equal to *a' e'* determine the elevation of that face of the prism which is parallel to *a b g h*. The remaining four points must, by the symmetry of the figure, lie on a line parallel to the vertical trace *o h'* and bisecting *a' e'*, they can, therefore, be obtained from their plans by simple projection. A second elevation of the prism on a ground line *z z* is shewn, but needs no explanation.

Note *a*. Other methods of completing the projections of the prism after the determination of *a b g h* the plan of one face, might have been adopted. Thus a complete elevation on a ground line at right angles to *t p* might have been readily drawn, as in Prob. III. of this Chapter, and the plan thence deduced.

Note *b*. If the two planes whose inclinations are given make any angle θ with each other instead of being mutually at right angles, as in the above case, the horizontal trace of the second plane must be determined by means of Prob. XVI., Chap. III. This horizontal trace and the intersection of the two planes being first determined, the remaining construction is precisely similar to that above explained.

Projections of cones and cylinders.

The determination of the projections of right cones and cylinders should present no difficulty, since it depends on the projection of circles explained in Chap. III., *p.* 39. It should be noticed that in the case of a cone whose axis is inclined to the plane of projection, the projection of the vertex and also of the base being determined, then the latter is an ellipse, and tangents to that ellipse from the projection of the vertex complete the outline of the projection of the cone. Similarly, in the case of a right cylinder, if the projections of the two ends are determined, tangents to those projections (ellipses) complete the outline of the solid. Two examples are added, but no figures are given.

Problem VII.

The plan of the axis of a right cone makes an angle α with xy, the axis itself is inclined β. Determine the projections of the solid, the base to touch the horizontal plane.

Draw a line making α with xy; take this line as the plan of the axis. Parallel to this line draw a ground line *z z*, and on *z z* draw an elevation of the cone. The vertical plane of which zz is the ground line is parallel to the axis of the cone, and hence perpendicular to its base; therefore the elevation of the cone on this plane will be an isosceles triangle of which the base is equal to the diameter of the base of the cone and the perpendicular from the vertex on the base is equal to the axis, or height, of the cone. Further, one angle of this triangle will be in zz and the perpendicular will be inclined at β (or the base at $90^\circ - \beta$), to zz. Hence this elevation can be drawn. By projection, the plan of the vertex and the axes of the ellipse are obtained. Draw tangents to this ellipse from the plan of the vertex, thus completing the plan of the solid. The elevation on xy is readily deduced. (See note on Projection of Circles, *p.* 40.)

Note. Compare this construction with that of Prob. II. of this Chapter.

Problem VIII. **The plan of the axis of a right cylinder makes α with xy, the base of the cylinder touches the horizontal plane, and is inclined β. Determine the projections of the solid.**

As in the last problem, draw the plan of the axis and make an elevation on a vertical plane zz parallel to the axis. The elevation on this vertical plane will be a rectangle of which one side is equal to the diameter of the base of the cylinder, and the other to its height or axis. The first-named side of this rectangle will have one extremity in zz, and will make β with zz. This elevation can therefore be drawn. Thence the equal ellipses which are the plans of the two ends of the cylinder are obtained; tangents to these ellipses complete the plan of the cylinder. The elevation on xy is then deduced. (See note on Projection of Circles, *p.* 40.)

HELICES, OR SCREW CURVES.

Helices.

If a point moves round the surface of a cylinder with a uniform circular motion, and has at the same time a uniform motion in a direction parallel to the axis of the cylinder, Cylindrical helix.

it will trace on the cylinder a curve termed a "cylindrical helix."

If a square or rectangle moves round a cylinder with a uniform motion, one of its sides being always on the surface of the cylinder and its plane normal to that surface, and if it has at the same time a uniform motion parallel to the axis of the cylinder, then each of its four angular points will trace a cylindrical helix, and the solid generated by the square or rectangle will be what is termed a "square screw thread." Similarly, an equilateral or isosceles triangle moving round a cylinder generates a "triangular" or "V-screw thread." (In practice, of course, screw threads are formed by cutting helical grooves round and out of a cylinder, and not by attaching threads outside it.) If there is only one tracing square, rectangle, or triangle, the screw is said to be "single threaded." The distance parallel to the axis of the cylinder traversed by a tracing point in one revolution is called the "pitch" of the screw or helix. In single-threaded square screws, the pitch is equal to twice that side of the tracing figure which is parallel to the axis of the cylinder, the groove into which the corresponding or "female" thread of the nut fits has, therefore, a breadth equal to that of the projecting thread. In a single-thread V-screw the pitch is equal to the base of the tracing triangle. The diameter of a screw is that of the cylinder out of which it is cut.

Square screw thread.

Triangular screw thread.

Pitch.

Diameter of a screw.

Problem IX. **Draw the projection of a square-threaded screw given its diameter and pitch.**

Suppose the screw to stand with its axis vertical and the tracing figure to be a square; the semicircle $a\,3\,e$ (Fig. 46, Pl. XI.), described with centre o on $x\,y$, and radius equal to half the given diameter is its half plan. Make $a\,b$, set off from one extremity of the diameter $a\,o\,e$, equal to half the given pitch. Then a square constructed on $a\,b$ as side is the tracing figure of the thread. Divide the semicircle $a\,3\,e$ into any number of equal parts, say six;

Fig. 46, Pl. XI.

the points of division are numbered 1, 2, 3, 4, 5. These points are, therefore, successive positions of the *plan* of the tracing point *a* as it moves uniformly round the cylinder. Lines from *a* and *e* at right angles to *a e* give the elevation of the cylinder from which the thread is cut; on these lines set off lengths, from *a* and *e* respectively, equal to the given pitch, and divide these lengths into twelve equal parts (twice the number of parts into which the semicircle *a* 3 *e* has been divided) by lines 1,′ . . . ., 12′ parallel to *a e*. Then, in one complete revolution the point *a* moves up to the line 12′; and since the motion is uniform, in half a revolution *a* will move to 6′. Similarly, in one-twelfth of a revolution *a* will move up to line 1′. Hence when the plan of *a* is at 1, its elevation will be on the line 1′; and when at 2, on line 2′, and so on. Thus projecting from 1, 2, 3, &c., on to lines 1′, 2′, 3′, &c., the elevations of successive points of the helix are obtained. Points on the inner helix traced by the point *b*, are obtained in the same way, a smaller semicircle being described with radius *o b*, and similarly divided. The helices described by the other angular points of the square will be exactly the same as those traced by the points *a* and *b* respectively. In practice it is best to draw the helical curves on thin wood or cardboard, cut them carefully out, and then use the "templates" so formed to rule in the successive convolutions of the thread.

In Fig. 46 the thread alone is shewn for the sake of clearness, the cylinder on which it is traced being omitted. The figure represents, therefore, a spiral spring of square section. Note *a*.

The line *a b* describes a "helicoidal surface." (See Definition, p. 53.) Note *b*.

This screw thread is terminated at *r s* on plan, giving the rectangle *r′ s′* in elevation. Note *c*.

In drawing screw threads, the helical curves are usually denoted by straight lines unless their size or pitch is very great. Note *d*.

Screw threads may also be traced by a moving circle or sphere. Instances of this are spiral springs of circular section and the "worm" of a still. The projection of such threads is obtained by drawing first the projection of the helix traced by the centre of the circle or sphere, and then describing a number of circles with centres on this helix and Note *e*.

radius equal to that of the tracing figure. Curves drawn to touch all these circles complete the outline of the projection of the thread.

EXAMPLES.

1. Draw the projections of cube of 3″ edge when one face is inclined at 50° and one edge of that face at 30° (Prob. IV., Chap. V.).

2. Draw the plan of the same cube when two of its edges are inclined at 45° and 25°. Draw also an elevation on a plane parallel to one of the diagonals (Prob. V., Chap. V.).

3. Draw the plan of the same cube when two of its adjacent faces are inclined at 45° and 75° (Prob. VI., Chap. V.).

4. Draw the projections of the same cube when its diagonals are inclined at 40° and 25° (Prob. V., Chap. V.).

5. Draw the plan of the same cube when three of its angular points are at heights of 1″, 1·25″, and 2″ above the horizontal plane. Make an elevation on a plane parallel to one of the diagonals.

6. A sphere of 1·75″ radius rests on the horizontal plane. Draw the projections of a cube inscribed in it. One of the angles of this cube to be at a height of 2·5″ above the horizontal plane, and the ***plan*** of one of the three edges forming this angle to make 30° with *x y*.

7. The plans of three edges of a cube of 4·5″ diagonal meet in a point and make angles of 120°, 130°, 110°. Draw the plan of the cube.

8. A pentagonal-right pyramid has height 4″ and side of base 1·5″. Draw its plan when one face is horizontal (Prob. II., Chap. V.).

9. Draw the plan of the same pyramid when one slant edge is vertical. Make an elevation on a plane parallel to one of the sides of the base (Prob. II., Chap. V.).

10. Draw the plan of the same pyramid when the base is inclined at 38°, and one side of the base at 20° (Prob. IV., Chap. V.).

11. Draw the plan of a hexagonal right pyramid, side of base 1·25″, height 4″, when one edge of the base and an adjacent slant edge are inclined at 20° and 30° respectively. (Determine plan of triangular face by Prob. XV., Chap. III. Find the point in which a perpendicular dropped from the centre of the base would meet this face. Obtaining the true length of this perpendicular, the centre of the base is determined. Complete by symmetry of hexagon.)

12. The traces of a plane make 45° and 30° with *x y*. Determine the projections of a tetrahedron of 3″ edge resting on this plane. One of the edges in the plane to be inclined 20°. (Turn down plane about its horizontal trace. Make also an elevation on a plane perpendicular to this horizontal trace.)

13. An oblique pyramid stands on a hexagonal base of 1·5″ side, three of its adjacent triangular faces are inclined at 60°, 50°, 40°. Draw its plan and an elevation on a plane making 25° with one of the sides of the base. (The intersection of the three planes whose inclinations are given is the vertex of the pyramid.)

14. A pentagonal prism (side of base 1·5″, height 4″) has its base inclined at 80° and one of its rectangular faces at 50°. Draw its plan. (Prob. VI., Chap. V.)

15. Draw the same prism when two of its adjacent rectangular faces are inclined at 75° and 30°. (Note *b*, Prob. VI., Chap. V.)

16. An oblique prism stands on a regular hexagonal base (side 1·25″), its length is 4″ and its axis makes 60° and 20° with the horizontal and vertical planes respectively. Draw its projections. (See Prob. III., Chap. II.)

17. Two adjacent edges (length 3″) of an octohedron are inclined at 37° and 24°. Draw the projections of the solid.

18. Two adjacent faces of the same octohedron are inclined at 28° and 78°. Draw its plan. (See Note *b*, Prob. VI., Chap. V.)

19. The plan of the axis of a right cone (height 4″, diameter of base 2·5″) makes 30° with xy. The axis is inclined at 50°. Draw the projections of the cone (Prob. VII., Chap. V.).

20. Draw the projections of the same cone when the plan of the axis makes 45° with xy, and one generating line is vertical. (Make an auxiliary elevation on a plane parallel to the axis, and then deduce the plan.)

21. A right cylinder (height 4″, diameter of base 2·25″) has its axis inclined 55° and 25° to the horizontal and vertical planes respectively. Determine its projections. (See Prob. III., Chap. II.; and Prob. VIII., Chap. V.)

22. Draw the projection of a square-threaded screw 3″ diameter and 0·75″ pitch (Prob. IX., Chap. V.).

23. Draw the projection of a V-threaded screw 3·25″ diameter and 1″ pitch. Tracing triangle isosceles with vertical angle 55°.

24. Draw the projection of a spiral spring 3″ diameter and 1·5″ pitch. Tracing figure a circle 0·75″ diameter. (See Note *e*, Prob. IX., Chap. V.)

CHAPTER VI.

SECTIONS OF SOLIDS BY PLANES.

Definition. Section.

THE section of a solid by a plane is a plane figure bounded by the lines (straight or curved) in which that plane cuts the surfaces (plane or curved) by which the solid is contained. The *form* of the section will depend upon the containing surface, or surfaces, of the solid and also upon the position of the latter with respect to the cutting plane. Thus the section of a pyramid is always a rectilinear figure, while the section of a right cone is (1st) a triangle; (2nd) a circle; (3rd) an ellipse; (4th) a parabola; (5th) a hyperbola, according as the cutting plane (1st) passes through the vertex; (2nd) is at right angles to the axis; (3rd) is inclined to the axis at a *greater* angle than the generatrix; (4th) is inclined at the *same* angle as the generatrix; (5th) is inclined at a *less* angle than the generatrix.

Sectional elevation.

The *sectional elevation* or *sectional plan* of a solid is its section combined with a projection on the section plane of those portions of the solid which would be visible from that plane.

To determine a section or sectional elevation of a solid, a plan and an elevation of the latter will usually be required.

The following problems will serve to explain the method of obtaining sections of solids by planes in various positions.

Sections by vertical planes.

SECTIONS BY VERTICAL PLANES.

Problem I.

Given the projections of a dodecahedron, to determine its section by a given vertical plane.

Fig. 47, Pl. XII.

In Fig. 47, Pl. XII., the projections of a dodecahedron, with one face on the horizontal plane, are shewn. A B is the ground line of the section plane.

The required section (which must of course be supposed turned down into the plane of the paper) might be constructed on A B, or any line parallel to A B as ground line; it may, however, conveniently be transferred to *x y*. The points 1.....7 are the *plans* of the intersections of the plane A B with the several edges of the solid, and by projection their elevations 1′.....7′ are obtained. These elevations determine the *heights* of the points at which the plane cuts the several edges of the solid (Theor. 4, *p.* 17). Hence, to construct the section required it is merely necessary to transfer the points 1.....7 to *x y* (preserving of course their proper intervals), and then to raise perpendiculars equal to their respective heights. This may readily be effected by drawing lines from 1′, 2′, &c., parallel to *x y*, cutting the respective perpendiculars. The seven-sided polygon thus obtained is termed the "section" of the dodecahedron "on A B."

Sectional elevation. If a sectional elevation is required, it is merely necessary to drop perpendiculars on A B from those points of the solid which would be visible from the section plane looking in the direction shewn by the arrows, to transfer to *x y* the points in which these perpendiculars cut A B simultaneously with the points 1.....7, and then to raise perpendiculars from the transferred points equal in length to the heights of the points themselves, obtained from their respective elevations. The manner of joining the points thus obtained with the angles of the seven-sided section can be at once seen by an inspection of the plan. The resulting figure is termed the "sectional elevation" of the solid "on A B;" it is shewn in Fig. 47, the sectional part being distinguished by cross hatching.

Note *a*. In practice, the points should be transferred from the section line to the new ground line by ticking them off with a fine-pointed pencil on the clean edge of a slip of paper.

Note *b*. The seven-sided polygon 1′ . . . . . 7′ is the *elevation of the section* of the solid.

In Engineering Drawing the term "section" usually implies "sectional elevation."

SECTIONS BY HORIZONTAL PLANES.

Sections by horizontal planes.

The section of a solid by a horizontal plane is obtained by a similar operation to that described in the last problem. A line parallel to *x y* and cutting the *elevation* of the solid will be the vertical trace of the section plane, and by projecting from the points in which this vertical trace cuts the elevations of the several edges of the solid on to the *plans* of those edges respectively, the horizontal section of the solid is obtained. Thus a horizontal section is determined by projecting from elevation to plan instead of from plan to elevation, as in the case of a vertical section. It should be noticed that the former of these processes always gives the *true form* of the horizontal section, while the latter gives the true form of the vertical section only when the section plane is taken parallel to the vertical plane of projection.

Fig. 48, Pl. XII.

In Fig. 48, Pl. XII., an octohedron with one face in the horizontal plane is shown in plan and elevation (see Prob. V., Chap. V.). This solid is cut by three horizontal planes dividing its height into four equal parts. The lines 1′ 1′, 2′ 2′, 3′ 3′, are the vertical traces of these planes, while the six-sided figures 11...&c., 22...&c., 33 ... &c., are the sections of the solid by the three planes respectively. All the points, 1, 2, and 3, can be determined by projection from their elevations; but since the intersections of some of the projecting lines with the plans of the edges of the solid would be very acute, these points are best determined by dividing the plans of the six inclined edges of the solid into four equal parts and joining the points of division all round the solid.

Contours. Definition.

When a solid or surface of any form is cut by a series of equidistant horizontal planes, the *plans* of the resulting sections are termed "contours" of the solid or surface. Thus the three six-sided polygons shewn in Fig. 48 are contours of the octohedron, and the contour of any solid contained by planes is made up of horizontals (see p. 45) of those planes. In Fig. 48, those portions of the contours which are hidden by the solid are shewn in dotted lines. If the upper portion of the octohedron cut off by one of the section planes ($1'\,1'$) be supposed removed, the remaining figure would be the "sectional plan" of the solid, or "plan at $1'\,1'$."

Sectional plan.

SECTIONS BY OBLIQUE PLANES.

Sections by oblique planes.

When a solid is contained by planes, the projections of its section by any plane given by its traces can be obtained by determining separately the projections of the intersections of the plane with each edge (Prob. VI., Chap. III.). In many cases, however, it will be found simpler to assume a new ground line at right angles to the horizontal trace of the given plane, to draw the elevation of the solid and vertical trace of the plane on the new co-ordinate plane, and then to project from the points in which the new vertical trace cuts the several lines of the new elevation. The following problem is an example of this method of proceeding.

Problem II.

An oblique pyramid stands on an irregular five-sided base; determine the projections of its section by a plane inclined to both planes of projection.

Fig. 49, Pl. XII.

In Fig. 49, Pl. XII., $a\,b\,c\,d\,e$, $a'\,b'\,c'\,d'\,e'$ are the projections of the base, and v', v those of the vertex of the pyramid; $h\,O\,t$ is the given plane. A new ground line $z\,z$ is assumed anywhere at right angles to $O\,t$ the horizontal trace of the given plane, and cutting it in s. Determine the inclination (a) of the given plane (Prob. III., Chap. III.); then a line $s\,l$ from s, making a with $z\,z$, will be the new vertical trace of the given plane (Theor. 26, *p.* 26). Make a complete elevation of the pyramid on $z\,z$. Then the point in which $s\,l$ cuts the elevation of any edge of the solid is the elevation (on $z\,z$) of one of the angular points of the required section. Thus i'' is the point in which $s\,l$ cuts the edge $v''\,a''$, and its plan i (obtained by projecting from i'' on to $v\,a$) is one of the angular points of the required plan of the section of the pyramid by the plane *h* O *t*. The remaining points of the plan are similarly obtained, and by projection their elevations on *x y* are deduced.

The upper portion of the pyramid cut off by the plane *h* O *t* has been removed in the figure, and both the projections of the section are distinguished by cross hatching.

Note. The true form of the section can readily be obtained by rotating it down about O *t*, the horizontal trace of its plane. (See Note *d*, Prob. XIV., *p*. 36.)

Sections of curved surfaces.

SECTIONS OF CURVED SURFACES BY PLANES.

Right cylinder. If the surface is a right cylinder, its section by any plane oblique to the axis will be an ellipse. If the right cylinder be enveloped by a square prism, and the section of the prism by the plane be determined, the resulting figure will be a parallelogram circumscribing the required ellipse. Right cone. Similarly, if a right cone be enveloped by a square pyramid, the section of this pyramid by a plane oblique to its axis will be either a quadrilateral or a figure of three lines not closed. In the former case, the section of the right conical surface will be the ellipse inscribed in the quadrilateral. In the latter case the section will be either a parabola or a hyperbola, and the three lines will be three tangents to the curve. Further, if the intersections of the cutting plane with those generatrices in which the pyramid touches the conical surface be determined, these intersections give points on the curve, which can therefore be drawn. The following general method is applicable to *any* conical or cylindrical surface, provided only that its trace (see Definition, *p*. 37) is given or can be determined.

Problem III. **A conical surface has a circular trace; this trace and the vertex are given. Determine, 1st, the projections of the section of the surface by a plane with given traces; 2nd, the true form of this section.**

Fig. 52, Pl. XII. In Fig. 52, Pl. XII., *v v′* is the vertex of the conical surface, and the circle with centre *c* is its horizontal trace; *mo*, *on* are the traces of the plane. (Lines drawn at right angles to *x y* and touching the trace of the surface determine its elevation.) Then, if the plans of the intersections of a sufficient number of generatrices with the plane be determined, the required plan of the curve can be drawn through them. In the figure, *v*′ 1′ is the common elevation of two generatrices *v* 1 and *v* 2. The plane *m o n* cuts these generatrices in the points *a*′ in elevation, and *a*, *b* respectively in plan; *a* and *b* are therefore points on the required plan of the section of the surface. Proceeding in this way, a sufficient number of points can be found to enable the plan of the section to be drawn.

The *true form* of the section is obtained by rotating it down about the horizontal trace (*o n*) of its plane. Thus making *a* A, *β* B, each equal to *o a*′, we obtain A and B, points on the turned down or true section.

The plan of the section is distinguished in Fig. 52 by cross hatching, and the upper portion of the surface cut off by the plane *m o n* is supposed to be removed on plan. Note *a*.

The cutting plane *m o n* has been assumed perpendicular to the vertical plane of projection. If this were not the case, the above construction could still be applied after taking a new ground line and making a new elevation as explained in Prob. II. of this Chapter. Note *b*.

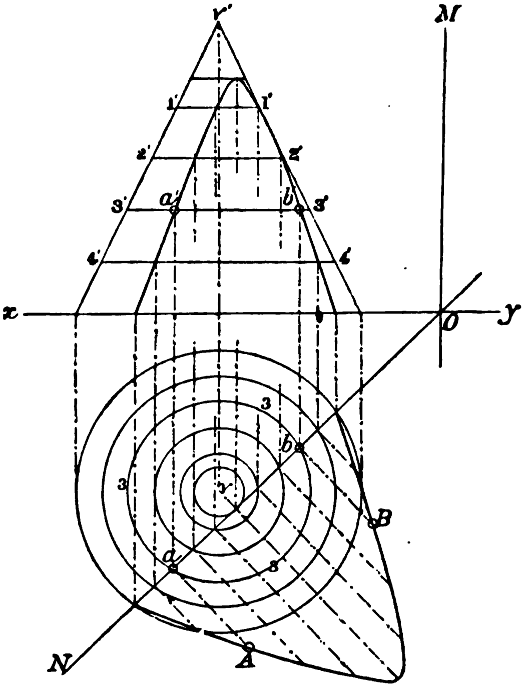
In the case of a cylindrical surface given the axis and trace, the construction would be similar, but the generatrices would be *parallel* in plan and elevation respectively, instead of converging. Note *c*.

No absolute rules can be laid down applicable to all kinds of surfaces, but it may be remarked that surfaces will usually contain certain sets of lines or curves easily obtained, and the intersections of such lines or curves with the section plane determine points on the section of the surface. Thus, in the case of all solids of revolution (Def. *p*. 52), sections by planes at right angles to the axis are circles, and the points in which these circles are cut by any plane will be points on the section of the solid by that plane. This is illustrated in the case of the right cone in the following problem.

Problem IV. **Determine the section of a right cone standing on the horizontal plane: 1st, by a vertical plane; 2nd, by any oblique plane.**

Fig. 50. 1st. The given right cone (Fig. 50) is cut by the vertical plane M O N. If now the cone is supposed to be cut by a series of horizontal planes, the sections (or contours) will be circles and their plans will be concentric circles, with the plan of the vertex of the cone as centre. Draw any series of lines parallel to xy cutting the elevation of the cone in 1′ 1′, 2′ 2′. . . &c. Then 1′ 1′, 2′ 2′ . . . are the elevations of the circles in which horizontal planes at these respective levels cut the cone. Draw their plans, and from the points in which N O, the horizontal trace of the given

FIG. 50.



plane, cuts each of them, project up on to the corresponding elevation. Thus, from the points *a*, *b* on the circle 3 3 3, project on to the line 3′ 3′. Then a' and b' are points on the elevation of the curve. The other points are similarly determined.

The real form of the section is obtained by turning the latter down about O N. Thus *a* A and *b* B are made equal to the height of the elevations $a,' b'$, and so on. The section is a hyperbola.

In the figure the horizontal cutting planes have been taken as equidistant, but they can of course be taken anywhere at pleasure. **Note *a*.**

To determine the precise elevation of the vertex of the hyperbola, draw a circle with *v* as centre touching O N. Obtain the elevation of this circle, and project up to it from the point of contact. **Note *b*.**

2nd. If the cutting plane is oblique, as in Fig. 51, the principle is exactly the same, but in this case the plan of the intersection of the cutting plane M O N with each of the contour planes must be determined. Thus *t s* is the plan of the intersection of the plane M O N with the horizontal plane cutting the cone in a circle whose elevation is 3′ 3′, and *t s* cuts the plan of that circle in *a*, *b*, two points on the plan of the required curve. By projection, a' and b' the elevations of those points are obtained, and similarly for all the other points. The curve is an ellipse, and the plans of its vertices can best be obtained by drawing an auxiliary elevation of the cone on a ground line *z z* at right angles to *x y*. $M_1 O_1$ making an angle with *z z* equal to the inclination of the plane M O N, is the new vertical trace of the plane, and it cuts the outline of the new elevation of the cone in c', d'. Project down from c', d' on to a line through *v* at right angles to N O. Then *c*, *d* are the vertices of the ellipse. The real form of the ellipse can be obtained by turning it down about N O as shewn, and in this the auxiliary elevation will assist. Fig. 51.

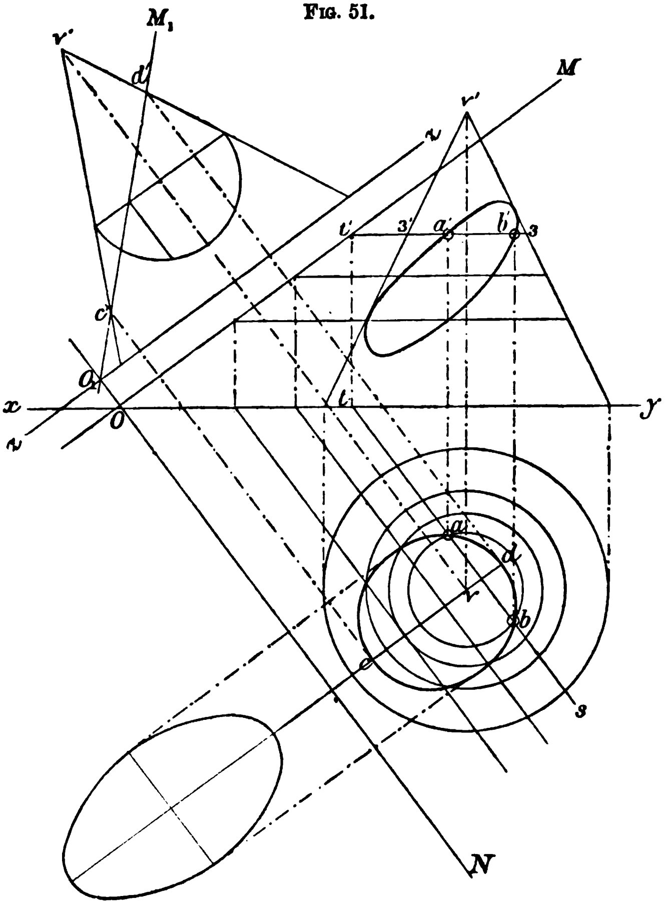
The above constructions are applicable to any solid of revolution with a vertical axis.

The curve of section being an ellipse, both the plan and elevation are ellipses also. Moreover one of the axes of the elliptic plan is parallel **Note *c*.**

and the other at right angles to the horizontal trace of the cutting plane.

Note *d.* $c'd'$ is equal to the major axis of the true ellipse. To determine the minor axis directly, bisect $c'd'$ by a line parallel to zz terminated by the

Fig. 51.



outline of the elevation of the cone. Describe a semicircle on this line, then the ordinate of this semicircle at the point of bisection is equal to half the minor axis. (See Prob. XVI., Chap. III.)

Note *e.* In the figure the cutting plane has been so taken as to give an ellipse, but the construction is perfectly general, and would apply equally if the section were a parabola or hyperbola.

EXAMPLES.

1. A right cylinder, diameter 2″, axis vertical, is cut by a plane inclined at 48°. The horizontal trace of the section plane touches the base of the cylinder and makes 55° with xy. Determine the elevation of the section and also its true form.

2. A right cone, base 3″ diameter, height 4″, stands on the horizontal plane. It is cut by a plane whose vertical and horizontal traces make angles of 35° and 60° with xy respectively. Determine the projections and true form of the section.

3. The same cone is cut by a plane parallel to a generatrix and bisecting the axis. The horizontal trace of this plane makes 45° with xy. Determine the projections and true form of the section.

4. The same cone is cut by a vertical plane which bisects a generatrix and has a horizontal trace inclined 45° to xy. Determine the elevation and true form of the section.

5. A sphere (radius 1·5″) resting on the horizontal plane is cut by a plane passing through its centre, inclined at 50° and having a horizontal trace making 60° with xy. Determine the projections of the section. Determine also the projections of the section of the sphere by a second plane parallel to and 75″ from the first.

6. One of the faces of a tetrahedron of 2·5″ edge is vertical and inclined 30° to the vertical plane of projection. One edge of this face is inclined at 20°. The solid is cut by a vertical plane bisecting any edge and cutting $\frac{1}{4}$ off an adjacent edge. Determine the projection and true form of the section.

7. A prism, length 3·5″, has an equilateral triangle of 2″ side as base. This prism is cut by planes into three equal pyramids. Show the lines in which these planes would cut the prism, and draw the plan of one of the pyramids when its plane of section is on the horizontal plane.

8. A cube of 3″ edge has one diagonal inclined at 70°. It is cut by equidistant horizontal planes dividing this diagonal into five equal parts. Show contours.

9. A pentagon 1·5″ side is the base of an oblique prism whose parallel edges make 55° and 20° with the horizontal and vertical planes respectively. Determine the projections of its section by a plane inclined at 45°, and cutting two adjacent parallel edges in points 25″ and 1″ from the base.

10. An ellipse with axes 3″ and 1·5″ is the horizontal trace of a cylindrical surface whose axis is inclined at 50° and 30° to the horizontal and vertical planes respectively. The major axis of the ellipse produced makes 35° with xy. The surface is cut by a plane inclined at 40°, whose horizontal trace touches the elliptic trace and makes 65° with xy. Determine the projections of the section.

11. Two horizontal lines are 1·5″ and 3·5″ above the horizontal plane, their plans make an angle of 95° with each other, and one of

them makes 40° with xy. A surface is generated by a line moving parallel to the vertical plane and meeting the two lines. Determine its trace.

12. An annulus is generated by a circle 2″ diameter, whose centre moves on a circle 3″ in diameter. The solid rests on the horizontal plane. Determine, 1st, the projection of its section by a vertical plane touching its inner surface and inclined 60° to the vertical plane of projection; 2nd, the projections of its section by a plane whose horizontal and vertical traces make 60° and 50° with xy respectively, and which also touches the inner surface of the annulus. (This solid is called an "anchor ring," and it possesses the following peculiar property: all planes which touch its surface in two points cut it in two equal intersecting circles.)

CHAPTER VII.

INTERPENETRATION OF SOLIDS, ETC.

SOLIDS CONTAINED BY PLANES.

THE following is a general method for determining the intersection of any two such solids, in any position with respect to each other. In many cases, however, special forms or positions of solids will suggest simpler methods of operation.

In Prob. VI., Chap. IV., the intersection of two planes is determined by means of their horizontals. If therefore two interpenetrating solids are cut by a series of equidistant horizontal planes, the "contours" obtained (see Definition, *p.* 72) will severally intersect in points on the plans of the intersections of the respective faces of the solids. That is to say, the intersections of any two contours (produced if necessary) of a face of one solid with the corresponding contours of a face of the other solid, determine the plan of the intersection of the planes of those faces. Hence, if any portion of this plan *falls within the plans of both the faces in question,* that portion will be common to both solids, and will therefore be part of their common intersection.

This method of using contours in determining the intersections of two bodies is of considerable importance. The following problem will serve as an example.

Problem I.

Determine the projections of the intersection of an oblique prism having an irregular four-sided base, with an oblique pyramid having an irregular five-sided base.

Fig. 53, Pl. XIII.

In Fig. 53, Pl. XIII., the projections of the two solids are shewn: *a b c d e* is the base of the pyramid; *f g h k* that of the prism. The lines 1′ 1′, 2′ 2′ are the vertical traces of two horizontal planes cutting both solids. The contours obtained will be, on the prism four-sided polygons with sides respectively parallel and equal to those of the base *f g h k*, and on the pyramid five-sided figures similar to *a b c d e*, and with sides respectively parallel. Thus, one point on each contour being obtained by projection, the contours can be completed by drawing lines parallel to the sides of the bases, and terminated at the plans of the edges of the solids. The contours thus determined are figured 1, 1, 1, 1, 1, and 2, 2, 2, 2, the same figure being used for the contours of similar level on both solids. Now the contours of face *b v c* of the pyramid and *g h* of the prism respectively intersect in the points *p* and *q*; the line $\beta\,\epsilon$ therefore drawn through *p* and *q*, and terminated by the plans of the edges of the face *b v c*, is the intersection of faces *b v c* and *g h*. Similarly, the contours of the faces *b v c* of the pyramid and *f k* of the prism respectively intersect in *s* and *t*; the line $\delta\,\gamma$ therefore drawn through *s* and *t*, and terminated by the plans of the edges of the faces in question, is the intersection of those faces. Proceeding in this way, the complete plan of the intersection of the two solids is obtained. The elevation of this intersection is deduced by projection; thus $\beta'\,\epsilon'$ is the elevation of $\beta\,\epsilon$, $\delta'\,\gamma'$ of $\delta\,\gamma$, and so on.

Note *a*. In the present instance two contours of each solid are sufficient: a larger number is however sometimes required, as will be seen in the case of curved surfaces.

Note *b*. In Fig. 53 the contours are shewn in fine continuous lines where they would be visible, and in fine dotted lines where they would be hidden.

Note *c*. The *form* of the intersection of these two solids should be noticed. The pyramid passes *completely through* the prism, and hence the intersection consists of two separate polygons. One of these polygons is a *plane* figure, since it lies wholly in the face *g h* of the prism; the other does not lie in any one plane, and can therefore be represented on paper only by its projections or development. If a portion of the pyramid (the edge *v b*, for instance) had fallen *outside* the prism, there would have been only one polygon, and this not plane.

Note *d*. In determining the intersection of two solids by cutting both by a system of parallel planes, it is not necessary that the latter should be horizontal. Any parallel planes may be used, but those whose intersections with the solids can be most readily determined, should of course be selected. For this reason vertical or horizontal planes are usually preferable.

Curved Surfaces.

The intersection of two curved surfaces may be either, 1st, *a straight line,* as in the case of two cylinders with parallel axes; or, 2nd, *a plane curve,* as in the case of a sphere and a right cone whose axis passes through its centre; or, 3rd, *a curve of double curvature,* as in the case of two unequal cylinders whose axes are not parallel.

The lines (straight or curved) in which two interpenetrating surfaces are cut by a plane or a third surface, will themselves intersect in points on the common intersection of the two surfaces. **Theorem 33.**

Hence, if everything is known with respect to the relative positions and modes of generation of two surfaces, points on their common intersection can always be obtained by cutting both surfaces by a system of planes or of surfaces. *That particular system of planes or surfaces whose intersection with the solids is most easily determined, should of course be selected for the operation.* Thus, two surfaces of revolution whose axes are vertical, should be cut by a system of horizontal planes. Two conical surfaces with oblique axes and any bases, should be cut by a system of planes passing through both vertices. A conical and a cylindrical surface should be cut by a system of planes containing the vertex of the first surface and parallel to the axis of the second. Two surfaces of revolution whose axes intersect should be cut by a system of concentric spheres, whose common centre is the point of intersection of the axes.

Determine the projections of the intersection of a right cone and right cylinder having axes respectively vertical and horizontal. **Problem II.**

The projections of the solids are shewn in Fig. 54, Pl. XIV., and it will be seen that a system of horizontal Fig. 54, Pl. XIV.

planes will cut the cone in a series of circles concentric on plan with its base, and the cylinder in a series of pairs of generatrices. To save construction the cutting planes are taken, not vertically equidistant, but so as to cut the cylinder in equidistant generatrices. Thus, the semicircular half end of the cylinder being revolved parallel to the vertical plane about $1' 1'$ the major axis of its elliptic projection, is divided into six equal parts in the points $2'$, $3'$, $4'$, $3'$, $2'$. Horizontal lines through these points are the *elevations* of the sections of the cylinder by the system of horizontal planes. The *plans* of these lines are found by turning down the half end of the cylinder, dividing it into six equal parts as before, and drawing lines through the points of division parallel to the axis of the cylinder. The centre line is numbered 1, and the remaining lines are numbered 2, 3, 4 ; 2, 3, 4 outwardly from it. It will be seen, therefore, that each line in elevation, except those numbered $1'$, corresponds to two lines on plan; and similarly each line on plan, except those numbered 4, corresponds to two lines in elevation. Now, the plane which cuts the cylinder in the lines of which $4'$ is the elevation and 4, 4, the plans, will cut the cone in a circle of which $d' d'$ is the diameter. Hence, if with v (the centre of the base of the cone) as centre and $d' d'$ as diameter, or $d' o'$ as radius, a circle is described, the points e, e, e_1, e_1, in which this circle cuts the lines 4, 4, will be points on the required plan of the intersection of the solids. Points on the lines 1, 2, and 3 are similarly obtained, and the plan of the intersection is drawn to pass through them. The elevations of the points thus determined on plan are obtained by projection. Thus e', e'_1, e', e'_1, on line $4'$, are the elevations of e, e_1, e, e_1. An inspection of the plan shews which of the points of the curves are on the *lower half* of the cylinder, and the elevations of these points must be on the *lower* lines similarly numbered.

Note *a*. The curves of intersection obtained in this case are of "double curvature"; that is to say, they partake of the curvature of both solids, do not lie in any one plane, and cannot be represented on paper except by their projections. They are, however, capable of development in two ways, as will be seen hereafter.

Note *b.* Since the cylinder passes completely ***through*** the cone, there are two separate curves of intersection. If the cylinder were partially ***outside*** the cone, there would be but one curve. If the cylinder were to ***touch*** the cone internally, the curve of intersection would have two coincident cusps.

Note *c.* It should be noticed that the bounding lines of the solids ***touch*** the curves of intersection both in plan and elevation.

Note *d.* In the figure the unseen portions of the curves are dotted, and the bounding lines of the one solid are not shewn where they are cut away by the other solid.

The above is a good example of the simple intersections which occur in drawing out machine details.

Problem III.

Determine the projections of the intersection of two cylindrical surfaces whose axes are oblique to both planes of projection, and do not meet.

Fig. 57, Pl. XV.

In Fig. 57, Pl. XV., the projections of two right cylindrical surfaces are shewn. The traces of these surfaces are ellipses which, if not given, must first be determined. Then if the two surfaces are cut by a system of planes *parallel to both axes*, these planes will cut the surfaces in generatrices whose intersections will give points on the common intersection of the surfaces.

A point $p\,p'$ is assumed on the axis of one cylinder, and from it a line is drawn parallel to the axis of the other cylinder. Then $h\,t_1$, $h\,t_2$, being the horizontal traces of this line and the axis of the former cylinder respectively, $z\,z$ is the horizontal trace of the plane containing them (Theor. 24, *p.* 26), and $s\,s$, $v\,v$ parallel to $z\,z$ will be the horizontal traces of planes parallel to the axes of both cylinders. The three lines $z\,z$, $s\,s$, $v\,v$, cut the elliptical traces of the cylinders in the points 1, 1, 1, 1; 2, 2, 2, 2; 3, 3, 3, 3 respectively; these points are therefore the *horizontal traces* of the generatrices in which the three planes cut the two surfaces. The *plans* of these generatrices are obtained by drawing lines through their horizontal traces

1..1, 2..2, 3..3, parallel to the axes of the respective cylinders. The intersections of these plans, the points *a*, *a*, *a*; *b*, *b*, *b*, *b*; *c*, *c*, *c*, are points on the plan of the common intersection of the surfaces. Similarly the intersections of the *elevations* of these generating lines give the points a', a', a'; b', b', b', b'; c', c', c', on the elevation of the required intersection. By taking more traces parallel to *z z*, and repeating the above construction, sufficient points are obtained to enable the required curves to be drawn.

Note *a*. Since one of the cylinders passes completely through the other, there will be two curves of intersection, one of which will be a closed curve, while the other will terminate at i_1 and i_2, the points in which the elliptical traces intersect, unless the cylinders are supposed to be produced through and below the horizontal plane of projection.

Note *b*. The cylinders shewn in Fig. 57 are right cylinders, the method explained for determining their intersections is however applicable to *any* cylindrical surfaces whatever, provided only that their traces are known or can be obtained.

Note *c*. When the axes of both cylinders are oblique to the planes of projection, as in this instance, it will usually be impossible at first to see what form the intersection will assume. This form will, however, become apparent when a sufficient number of points have been determined.

TANGENTS TO CURVES OF DOUBLE CURVATURE.

The limits of the present work do not allow of a full treatment of the methods of determining tangents to curves of double curvature. Two points in connection with the subject may, however, be noticed.

Theorem 34. 1st. The tangent to a curve of double curvature must be also a tangent to both the containing surfaces. Hence, if at any point on such a curve a tangent plane to each surface is determined, the projections of the intersection of these two tangent planes will be the projections of a tangent at that point to the curve of double curvature.

Theorem 35. 2nd. Tangents at any point to the projections of a curve of double curvature are the projections of the tangent to the curve itself at that point.

DEVELOPMENT.

If a surface can be folded out on to a plane, it is said to be "developable," or "capable of development," and the plane geometrical figure obtained by thus folding out a surface is termed its "development." **Definition of development.**

All surfaces generated by the motion of a straight line are developable if the generatrix in two successive positions always lies in one plane. Thus conical and cylindrical surfaces are always developable, while spherical, spheroidal, helicoidal and conoidal surfaces are incapable of development. **General rule.**

Surfaces generated by the motion of a straight line and incapable of development are termed "twisted surfaces." **Twisted surfaces. Definition.**

The helix, whether cylindrical or conical, is developable, since it is a curve of double curvature lying on a cylindrical or conical surface. Thus, referring to Fig. 46, Pl. XI., if *a d* is made equal to the perimeter of the semicircle *a* 3 *e*, and *d q* (at right angles to *a d*) to half the pitch of the helix, then *q* represents the point 6′ when the curve *a* 6′ is developed or folded out on to the vertical plane of which *x y* is ground line. Further, since the helix has a uniform twist, the ratio existing between the ordinates of the developed curve is constant, and the developed helix is thus the straight line joining *a* and *q*. The angle *q a d* is called the "inclination of the thread." Similarly, *b p* is the development of the inner helix, and the angle *p b c* its inclination. **Development of the helix. Fig. 46, Pl. XI.**

If the helix is not uniform but has an increasing twist, as is the case of the rifling of some heavy guns, the ratio between the ordinates of the developed curve is not constant, and the latter therefore becomes a curve. **Note.**

The curves of double curvature represented by their projections in Fig. 54, Pl. XIV., lie on two surfaces—a right cylinder and a right cone—these curves are therefore capable of development in two ways. **Fig. 54, Pl. XIV.**

Fig. 55, Pl. XIV. Development of right cylinder.

First. If cf, Fig. 55, Pl. XIV., is made equal to the perimeter of the circular end of the cylinder shown in Fig. 54, and divided into twelve equal parts, then ordinates drawn from all the points of division at right angles to cf represent the generatrices of the cylinder when developed or folded out on to the horizontal plane. Hence, if these developed generatrices (Fig. 55) are made respectively equal to their corresponding plans (Fig. 54), from the end of the cylinder to the points where those plans meet the plan of the curve of intersection, points on the development of the latter are obtained. Thus, the two lines $1''\ 6''$ (Fig. 55) are both made equal to 1 6 (Fig. 54)—since these lines will coincide if the developed cylinder be supposed reconstructed—and each successive generatrix is made equal to its corresponding plan, working completely round the cylinder, $2''\ 8''$ equal to 2 8; $3''\ 9''$ equal to 3 9, and so on.

Note *a*. It is only because the cylinder (Fig. 54) is horizontal that the developed generatrices can be made equal to their *plans* (see Theor. 12, *p.* 18); if the cylinder were inclined, the *real lengths* of these generatrices must be determined.

Note *b*. As drawn in Fig. 47, the cylinder has been supposed ripped up along the line 1 6 and then folded flat. It is obvious, however, that any other line might have been taken instead.

Note *c*. The two intersecting cylinders given in Fig. 57, Pl. XV., with their curve of intersection, could both be developed in the same way. The latter would take a different form for each of the developments.

Development of right cone.

Second. If the cone be supposed cut in halves by a vertical plane $z_1\,z_2$ passing through its axis, the development of one of the halves would be a sector of a circle having a radius equal to the slant height of the cone and an arc equal to half the perimeter of its base. $V\,Z_1\,Z_2$ (Fig. 56) is thus the development of the half cone, the length of arc $Z_1\,Z_2$ being obtained either by dividing up $z_1\,z_2$ into small arcs and setting off the chords of these arcs along the arc $Z_1\,Z_2$ (thus assuming the chord of a small arc whose radius is $v\,z_1$, to be equal to the chord of an equal arc whose radius is $V\,Z$, a sufficiently close approxi-

Fig. 56, Pl. XIV.

mation), or else by calculating the angle $Z_1 V Z_2$. It is now required to place in this sector the development of one of the curves of intersection of cylinder and cone. Draw *v a*, *v b* (Fig 54) touching the plan of the curve, make the arcs $Z_1 a'$ and $Z_2 b'$ (Fig. 56) equal to $z_1 a$ and $z_2 b$ respectively, then the lines V *a*′, V *b*′ will touch the developed curve. Divide the arcs *a b* and *a*′ *b*′ into any number of equal parts, say six, and draw lines from *v* to all the points of division (3, 4, 5, 6, 7) of the arc *a b*, and from V to all the corresponding points (3′, 4′, 5′, 6′, 7′) of the arc *a*′ *b*′. The former lines cut the plan of the curve of intersection in points whose *real distances* from the vertex of the cone, set off along the corresponding lines of Fig. 56, give points on the required development. Thus the line *v* 4 (Fig. 54) cuts the plan of the curve in the points *r* and *s*. Determine the real lengths of which *v r* and *v s* are the plans, and set off these real lengths on the line V 4′ (Fig. 56) from V, thus obtaining S and R points on the development required. Points on the lines V 3′ V 5′, &c., can be determined in a similar manner.

Note.

The real lengths of *v r*, &c., may conveniently be found as follows. Make *o ρ* equal to *v r*, and draw *ρ ρ*′ from *ρ* at right angles to *x y* and meeting *v*′ *n*′, the bounding line of the cone in *ρ*′, then *v*′ *ρ*′ is the real length of *v r*, and V R must be made equal to *v*′ *ρ*′.

Solids contained by planes.

The bounding surface of any solid contained by planes is of course always developable, and if two such solids intersect, the development of the lines of intersection can be obtained on the development of each. Thus, the prism and pyramid shown in Fig. 53, Pl. XIII., can both be developed together with the polygons (one plane and the other not) in which they intersect. The determination of such development needs no explanation, since it is merely necessary to find the real lengths of certain lines whose projections on two planes have been previously obtained. (See Prob. II., Chap. II.)

Definition.

The bounding line of the projection (i. e. the *apparent* *outline*) of a solid, is sometimes spoken of as its "contour." The term contour used in this sense must not be confounded with the section of a solid by a horizontal plane (Definition, *p.* 72).

Shortest distance between points on a solid.

The shortest distance between any two points on a solid contained by planes, or by a curved surface, will be the distance between those points developed. To find this distance, therefore, develope the solid. Determine the position of the points on the development, and join them.

EXAMPLES.

1. An octagonal and a heptagonal right pyramid stand on the horizontal plane, the former has height 4″ and side of base 1·5″, the latter 5″ and 1·25″ respectively. One angle of the base of the former is at the centre of the base of the latter. Determine the projections of the intersection of the two solids.

2. A vertical prism on an irregular five-sided base (no side less than 1·25″) stands on the horizontal plane, a horizontal prism with square base (1·75″ side) passes partially through the vertical prism. The lengths of both prisms are 4·5″; the long edges of the latter make 35° with the vertical plane, and one of these edges is outside the former prism and ·5″ from its nearest vertical edge. Determine the projections and development of the intersection of the prisms.

3. A vertical right pyramid and a horizontal right prism intersect. The former has a pentagonal base of 1·5″ side, the latter a square base 1·5″ side; the height of the former is 4·5″. The axis of the prism makes 40° with the vertical plane and cuts the axis of the pyramid at a height of 2″ above its base. Determine the projections of the intersection of the solids, and obtain the development of the pyramid.

4. A dodecahedron of 1·25″ edge (one face horizontal) is penetrated by a vertical pentagonal right pyramid, height 5″, side of base 1·5″. The axis of the pyramid cuts the horizontal faces of the dodecahedron at ·5″ from their centres, and the lower of these faces is 1″ above the base of the pyramid. Determine the projections of the intersection of the solids.

5. A vertical pentagonal right prism, side of base 1·5″, passes completely through a sphere of 2″ radius. The axis of the prism passes at a distance of ·25″ from the centre of the sphere. Obtain the elevation of the intersection of the solids on a vertical plane making 30° with one face of the prism.

6. A vertical hexagonal right pyramid, side of base 1·5″, pierces a sphere of 2″ radius. The height of the pyramid is 4·5″, its axis passes at ·25″ from the centre of the sphere, and this centre is 2·5″ above the base of the pyramid. Determine the projections of the intersection.

7. A vertical right cylinder, radius of base 1·25″, is cut by a plane inclined at 50°. Obtain the development of the curve of section. (This development is the projection of a cylindrical helix of long pitch.)

8. A hollow sphere, external radius ·2″, thickness ·5″, has a cylindrical hole bored through it. The diameter of the latter is 1·5″, its axis is inclined 40° to the vertical plane, and is ·75″ above the centre of the sphere. The boring cylinder *touches* the internal spherical surface. Determine the projections of the sphere.

9. A vertical right cone, radius of base 1·5″, height 4·5″, penetrates a sphere of 1·75″ radius. The centre of the sphere is 2″ above the base of the cone, and the latter touches the surface of the sphere. Determine the projections of the intersection, and obtain the development of the cone.

10. Two hollow spheres, external radii ·2″ and 1·5″, thickness ·5″, intersect. The surface of the larger sphere passes through the centre of the smaller, the *plans* of their centres are 1·25″ apart. Determine the plan of the intersection.

11. Two equal cylinders intersect; their diameters are 1·25″, and their axes are horizontal and vertical respectively. The axis of the former cylinder makes 35° with the vertical plane, and its bounding generatrix on plan passes *outside* the circular plan of the vertical cylinder and ·25″ from it. Determine the elevation of the resulting intersection and develop either of the cylinders.

12. A hollow vertical right cylinder, external diameter 2·5″, thickness ·5″, is pierced by a horizontal right cone, diameter of base 2·25″, height 4·5″. The axis of the cone makes 40° with the vertical plane, and the cone itself *touches* the internal cylindrical surface. Determine the projections of the intersection, the cone being supposed removed.

13. The horizontal traces of two right cones are ellipses whose axes are 4″, 2·5″ and 3·5″, 2·5″ respectively; the major axis of the former makes 25°, and of the latter 35° with $x\ y$. These elliptical traces touch each other. Determine the projections of the intersections of the cones.

14. Two equal right cones intersect, their heights are 4·5″, diameters of bases 3·5″, and axes respectively horizontal and vertical. The height of the horizontal axis is 2″; it is inclined at 40° to the vertical plane, and passes at a distance of ·25″ from the vertical axis. The former axis would be bisected *on plan* by a perpendicular dropped from the plan of the latter. Determine the projections of the intersection of the cones.

15. A circular annulus, or anchor ring, is traced by a circle of 1·5″ diameter whose centre moves on a circle of 3″ diameter. The plane of the latter circle is horizontal. The annulus is penetrated by a vertical right cylinder of 1″ diameter, which touches the annular surface. Determine the projections of the intersection.

16. The same annulus is cut by a vertical right cone, diameter of base 1·5″, height 3″. The plane containing the path of the centre of the tracing circle of the annulus bisects the axis of the cone, and the plan of this axis is situated ·5″ *within* the bounding circle of the plan of the annulus. The vertical plane containing the axes of the two solids makes 50° with the vertical plane of projection. Determine the projections of the intersection, and obtain the development of the cone.

CHAPTER VIII.

TANGENT PLANES TO SURFACES.

Definition. A PLANE is said to touch a surface at any point when it contains all tangents to the surface at that point.

If, therefore, the point on a surface at which a plane is to touch that surface is given, the plane cannot be made to fulfil any other condition.

A tangent plane to a sphere or spheroid and to all surfaces of revolution generated by *curved* lines can, if the point of contact is not given, be made to fulfil *two* other conditions. In the case of a *ruled surface*, however, since a tangent plane must, from the nature of the surface, contain one of the generatrices, it can only be made to fulfil *one* other condition.

Thus, a plane can be determined to contain two given points and touch a given sphere or spheroid, while a plane to touch a given cone or cylinder can be made to contain only one given point.

Note. It should be remarked, however, that in the case of a *twisted surface*, although the tangent plane at any point must contain the generatrix passing through that point, this plane will nevertheless be tangent to the surface *at that point only*, and may *cut* the surface on either side of the point of contact.

The following are examples of the determination of tangent planes to the sphere, right cone, and right cylinder. Since a representation on one plane of projection suffices to determine these surfaces, the method of indices or figured plans explained in Chap. IV. is adopted.

Problem I. **Determine planes to touch two given spheres and contain a given point.**

If the spheres are external to each other there will be four planes fulfilling the required conditions; if they touch each other externally, three planes; if they cut, two planes; if they touch internally, one plane; while if one sphere includes the other the problem is impossible.

Fig. 58, Pl. XVI. In Fig. 58, Pl. XVI., *a* 30, *b* 10 are the centres of the given spheres, *p* 30 is the given point, the unit of height being taken as 0·1″.

If a cone is determined enveloping the two spheres, the required planes must touch this cone. Similarly, if a second cone is determined with *p* 30 as vertex, and enveloping either of the spheres, the required planes must touch this cone also. If, then, the circles of contact of these two cones with the sphere are determined, the intersections of these circles of contact will be the points of contact of the required planes and the sphere. A plane therefore containing the vertices of the two cones and also one of the points of intersection of the circles of contact, will fulfil the required conditions.

In Fig 58, Pl. XVI., lines tangent to the circular plans of the spheres determine the plan of their enveloping cone, and *v* the meeting-point of these tangents will be the plan of the vertex. Since *v* lies on the line *a* 30, *b* 10 its index (– 2) can be determined. (Prob. I., Chap. IV.)

To determine the plan of the circle of contact of this enveloping cone, an elevation of the latter will be necessary. Using the line *a v* as a ground line of level 30, the elevation of the sphere on *a v* will coincide with its plan, and *v*′, obtained by making *v v*′ equal to 32 units (the difference of level of *v* and the ground line) will be the elevation of *v*. Tangents from *v*′ to the circular plan (now elevation) of the larger sphere determine the elevation of the enveloping cone, and *m*′ *n*′, the chord of contact of these tangents, will be the elevation of the circle of contact of the cone and sphere. Hence, an ellipse described with *m n* the plan of *m*′ *n*′ as minor axis, and with major axis equal to *m*′ *n*′, will be the plan of the circle of contact of the first enveloping cone and the larger sphere.

Again, since *p* the given point and *a* the centre of the larger sphere have the same index, the axis of the second enveloping cone will be horizontal, and its circle of contact vertical; hence *s t*, the chord of contact of tangents from *p*, will be the plan of this circle of contact, and *x* and *y*, the points in which *s t* cuts the elliptic plan of the first circle of contact, will be the points in which the tangent planes required will touch the sphere.

The indices of the points *x* and *y* can be obtained as follows: Suppose a great circle of the sphere passing through one of the points (*y*) to be rotated about its horizontal diameter till its plane is horizontal. This great circle will then coincide with the plan of the sphere, and y' (determined by drawing a perpendicular to *a y* from *y*) is the point *y* turned down or constructed. The distance $y\,y'$ is therefore the *difference of level* of *a* and *y*; and since *y* is below *a*, the index of *y* is $30 - y\,y' = 22$ units. Similarly, the index of *x* is $30 + x\,x' = 41$ units.

It remains merely to determine two planes, one containing the points *v*, *p*, *x*, the other the points *v*, *p*, *y* (Prob IV., Chap. IV.). The "scales of slope" of both these planes are shewn.

Note *a*. There is another right conical surface, which will envelope both spheres. This surface will have a vertex *between* the two spheres, and will envelope one of the latter in each of its *sheets*. By proceeding in exactly the same way with this second surface, two more tangent planes to the two spheres are obtained, completing the total number possible. The plan of this second surface and of *o* its vertex are shewn in Fig. 58, but the planes are not determined.

Note *b*. If the index of *p* had *not* been the same as that of *a*, the circle of contact of the second enveloping cone would have had an ellipse for its plan. The two ellipses would intersect in *four* points, but an inspection of the figure would shew which are the points in which the circles of contact really intersect.

Note *c*. The construction above explained could of course have been made on the smaller sphere.

Determine planes to contain a given point and touch a given right cone. Problem II.

Note. If the given point is *within* the conical surface the problem is impossible; if it is *on* the surface there will be *one* plane fulfilling the condition; if *without* the surface *two* planes.

Fig. 59, Pl. XVI. In Fig. 59, Pl. XVI., v 7, c 37 is the *plan* of the axis of the given cone, and using this line as a ground line of level 37, v' c is the *elevation* of the axis, and a' b' at right angles to v' c and equal to the given diameter of the base of the cone, is the *elevation* of that base on v c. From this elevation the elliptic plan of the base is obtained. (See Projection of Circles, *p.* 39).

Now the required planes must pass through the point v v', and if a line is drawn through v v' and p 19 (the given external point), they must contain this line. Further, if from the point in which the line through v v' and p 19 cuts the plane of the base of the given cone, tangents are drawn to that base, the required planes must contain these tangents, and must touch the cone in generatrices drawn through their points of contact. Determine p' the elevation on v c of p 19, and join v' p'.

Then v' p', produced, cuts the plane of the base in i'; hence by projection, i 41·4 is the *plan* of the point in which the line through v 7 and p 19 cuts the plane of the base. Lines from i tangent to the elliptic plan of the base of the cone are the *plans* of tangents to that base from the *real point* in which the line through the vertex and the given point cuts its plane. These tangents touch the base of the cone in points of which t_1, t_2 (the points of contact of their plans with the plan of the base) are the plans. The elevations t'_1, t'_2 of t_1 and t_2 (obtained by projection) determine the indices 46 and 31 of t_1 and t_2 respectively.

It remains merely to determine two planes, one containing v 7, i 41·4, and t_1 46, and the other containing v 7, i 41·4, and t_2 31 (Prob. IV., Chap. IV.).

The scales of slope of these planes are shewn in the figure.

Note *a.* The points t_1 and t_2 should be determined by one of the constructions of Plane Geometry.

Note *b.* The problem may be solved in another way without using the base of the given cone. If a sphere of any radius is inscribed in the given cone, and a second cone with the given external point as vertex is determined, also enveloping this sphere; then the required planes must touch both cones and can be determined as in Prob. I. of this chapter.

Problem III.

Determine planes to touch a given cylinder and be parallel to a given line.

Note.

If the given line is parallel to the axis of the given cylinder the problem is indeterminate. In all other cases there will be two planes fulfilling the required conditions, but if the given line *touches* the cylinder one of the planes must contain it.

Fig. 60, Pl. XVI.

In Fig. 60, Pl. XVI., *a* 27, *b* 5 is the plan of the axis of the given cylinder. On the line *a b*, taken as a ground line of level 5, an elevation ($d' l' h' g'$) of the cylinder is made. Again, *s* – 8·5, *t* 50·3 is the plan of the given line, and s' t' is its elevation on *a b*.

Now all planes tangent to the cylinder must be parallel to its axis; hence if a plane is determined *containing* S T the given line and parallel to A B the axis of the cylinder, the required planes will be parallel to this plane, and must therefore cut the plane of the base of the cylinder in lines parallel to that in which the plane containing S T and parallel to A B cuts it (Euc. XI., 16). Further, the lines in which the required planes cut the plane of the base must be tangent to the circle of that base (Theor. 31, *p.* 38); the *plans* of these lines will therefore be tangents to the elliptic plan of the latter.

From the point $t t'$ on the given line, a line is drawn parallel to A B the axis of the cylinder (see Prob. XIII., Chap. III.); t' i'_2 is the elevation and t i_2, the plan of this line, which cuts the plane of the base of the cylinder in i'_2 i_2. The given line (S T) cuts the plane of the base in i'_1 i_1. The indices of i_1 and i_2 (determined from their elevations) are 1·5 and 14·7 respectively. Then the line joining i_1 and i_2 is the plan of the intersection of the plane containing S T and parallel to A B with the plane of the base of the cylinder. Parallel to i_1 i_2 draw two lines *m m*, *n n*, touching the elliptic plan of the base of the cylinder in c and e respectively. Then the required planes will touch the given cylinder in generatrices whose plans $c\,d$ and $e\,f$ are drawn from c and e respectively. It remains therefore to determine two planes, one of which contains the lines $c\,d$ and $m\,m$, and the other the lines $e\,f$ and $n\,n$ (Prob. IV., Chap. IV.). The index of c (determined from its elevation c') is 11·8, and the indices of d and of any point on the line $m\,m$ can be obtained, since $c\,d$ is parallel to $b\,a$ and $m\,m$ to $i_1\,i_2$ (see Prob. II., Chap. IV.). The indices of e, of f, and of any point on the line $n\,n$ can be similarly obtained.

The scale of slope of the plane which touches the given cylinder in $c\,d$ is shewn in the figure; the other tangent plane is not determined.

Note *a*. In the construction above explained, the plane of the base (i. e. of one end) of the given cylinder has been made use of, and upon this plane the *traces* of the required planes have been determined. The construction is, however, applicable to any other plane cutting the cylinder: thus the *horizontal trace* of the given cylinder might have been obtained, and the *horizontal traces* of the required planes thence determined.

Note *b*. The points c and e, in which the lines $m\,m$, $n\,n$ touch the ellipse, should be determined by Plane Geometry. These points may, however, be obtained as follows:

Suppose the circular base of the cylinder to rotate about its horizontal diameter till its plane is horizontal. "Construct" the points $i_1\,i_2$ which lie in that plane (Note *d*, *p*. 36). Draw lines parallel to the line joining these constructed points and touching the turned-down circle of the base. Finally, *turn up* the points of contact of these tangents into the plane of the base again.

EXAMPLES.

(Unit 0·1″.)

1. The radius of a sphere is 1·5″, the index of its centre 30. Determine the scales of slope of planes tangent to this sphere at points whose indices are 39 and 20, and whose *plans* are 1·5″ apart.

2. Determine two planes touching the same sphere and inclined 50° and 40°. The horizontals of these planes to make 60° with each other.

3. A sphere, radius 1·75″, rests on the horizontal plane. A line whose inclination is 55° passes through its centre. Determine a plane

touching the sphere, parallel to the line, and containing a point whose index is 10 and whose *plan* is 2·25″ from that of the centre of the sphere.

4. The index of the centre of a sphere is 35, and its radius is 1·5″. A line whose inclination is 35° passes through a point whose *plan* coincides with that of the centre of the sphere, and whose index is 65. Determine planes containing the line and touching the sphere.

5. Three points whose plans are at the angles of a triangle of 3″, 3·5″, and 4″ sides, and indices − 5, 20, 35 are the centres of spheres of 1″, 1·75″, and 2″ radii respectively. Determine two planes touching all three spheres. (Eight such planes are possible.)

6. The plans of three points (indices 35, 45, 55) are at the angles of an equilateral triangle of 2·75″ side. Determine a sphere to contain the points and touch the horizontal plane.

7. A right cone (radius of base 1·5″, height 4·5″) *lies on* the horizontal plane. The plan of the centre of a sphere of 2″ radius is 3″ from the vertex of the cone. The plan of this sphere touches that of the cone, and the index of its centre is 40. Determine planes touching sphere and cone. (Four such planes are possible.)

8. The same cone stands upright on the horizontal plane. A sphere of 1·5″ radius has its centre on a level with the vertex of the cone, and at a distance from it on plan of 3″. Determine planes touching sphere and cone. (Four such planes are possible.)

9. The radius of the base of a right cone is 1·5″, its height is 4″. The index of the vertex is 52, and of the centre of the base 30. Draw a line through the vertex inclined at 50°, and making 50° *on plan* with the plan of the axis of the cone. Determine the two planes which contain the line and touch the cone.

10. Obtain the scales of slope of all planes tangent to the same cone and inclined 45°. (Four planes are possible.)

11. One extremity of the axis of a right cylinder (radius of base 1·5″, height 4·5″) is in the horizontal plane, the index of the other extremity is 33. The plan of the centre of a sphere of 1·75″ radius is distant 4″ from the plans of the extremities of the axis of the cylinder. The index of the centre of this sphere is − 20. Determine two planes touching cylinder and sphere. (Four planes are possible.)

12. Determine two planes touching the same cylinder and inclined 65°. (Four planes are possible.)

13. A line inclined 48° touches the same cylinder, its plan makes 45° with and bisects that of the axis of the cylinder. Determine the point of contact of the line and obtain the scale of slope of a plane containing it and touching the cylinder.

14. A spheroid is generated by the rotation of an ellipse about its major axis, which is vertical. The axes are 4″ and 3″, and the solid rests on the horizontal plane. Assume a point on the surface whose index is 37, and determine a tangent plane at this point.

15. Through a point (index 60) on the produced axis of the same

spheroid draw a line inclined 30°. Determine two planes containing the line and touching the spheroid.

16. An "Anchor ring," axis vertical, rests on the horizontal plane. The diameter of its tracing circle is 2″, and the centre of this tracing circle moves on a circle of 2″ radius. Through a point (index 50) on the produced axis of the ring draw a line inclined 45°. Determine planes containing the line and touching the Anchor ring. (Four planes are possible.)

CHAPTER IX.

THE DETERMINATION OF SHADOWS.

Use of shadows.

A *single* orthographic projection of an object is seldom sufficient to determine its real form. Two such projections on planes mutually at right angles, are required, and by comparing them a practised eye can form a true estimate of the form and relative dimensions of the object.

Suppose, however, that an object of which one projection has been obtained, is illuminated by rays of light proceeding in some known direction, and that the illuminated portions of the object and the shadows cast by its various parts upon each other and upon the planes of projection are determined and distinguished. Then, from the form and extent of these shadows, the eye will far more readily recognise the form and relative dimensions of the object, even though there is but one view of the latter. Similarly, if two orthographic views of an object are constructed, each with shadows properly determined and distinguished, it will be still more easy to realize the true form.

It will, therefore, be evident that in drawings required solely for descriptive purposes, and especially in drawings intended for eyes unpractised in reading and comparing orthographic views, the aid afforded by correctly delineated shadows is of great value.

OUTLINE OF SHADOWS.

General principles.

The rays of light are assumed to be all parallel, and if a solid in space is supposed to be illuminated and surrounded by these parallel rays, a certain portion of the latter will be intercepted. These intercepted rays will form either a prism or a cylindrical surface enveloping the solid; the former, if the solid is contained by planes; the latter, if by a curved surface. If then, this prism or cylinder of intercepted rays meets any surface whatever, it will trace on that surface the outline of the shadow cast by the solid.

The *line of contact* of this imaginary prism or cylinder with the solid which it envelopes, is the line separating the illuminated and unilluminated portions of that solid. If, therefore, the solid itself is removed, the line of separation (supposed to have a sensible breadth) alone remaining, the outline of the shadow cast by the latter on any surface would be identical with that of the shadow cast by the original solid. In the case of a solid contained by planes, the line of separation of light and shade is made up of those edges of the solid in which illuminated and unilluminated faces meet, and can usually be distinguished by inspection. In the case of solids contained by curved surfaces, this line of separation is more difficult to determine. On the cone or cylinder, however, the lines of separation will be generatrices, and on the sphere a great circle.

From the above it follows:—

1st. That the outline of the shadow cast by any plane figure (supposed opaque) on a plane parallel to itself is an *equal* figure.

2nd. That the outline of the shadow cast by any solid on any surface whatever must be *continuous*.

3rd. That the problem of delineating the outline of the shadow cast by any solid on any surface, reduces itself in general to the determination of the intersection of certain lines or planes with that surface.

Note *a*. If it appears uncertain whether of two plane faces of a solid both are in light or in shade, or one is in light and the other in shade, draw a ray through any point on their intersection, then if this ray passes *between the two faces*, both are in light or shade; if otherwise, one is in light and the other in shade.

Note *b*. Throughout this Chapter, as in Engineering Drawings generally, the rays of light are supposed parallel. If, however, they are supposed instead to proceed from a fixed point, the form then assumed by the intercepted portion of rays is that of a pyramid or conical surface. The principles of construction will, however, remain the same.

Determine the shadow cast by a vertical truncated right hexagonal pyramid on the horizontal plane of projection. Problem I.

Fig. 61, Pl. XVII. In Fig. 61, Pl. XVII., A....F is the base of the pyramid, and *a*....*f* the plan of its top. If then *r* and *r'* are the projections of a ray, the plans of all other rays will be parallel to *r* and their elevations to *r'* (Theor. 15, *p.* 19). The faces *b* C, *c* D, *d* E, evidently form the unilluminated portion of the solid. Hence, if the intersections with the horizontal plane (i.e. the ***horizontal traces***) of rays passing through the points *a*, *b*, *c*, *d*, are determined, the outline of the required shadow is obtained. The points β, γ, δ, ϵ, are these horizontal traces. It remains therefore merely to join B β, $\beta\ \gamma$, $\gamma\ \delta$, $\delta\ \epsilon$, ϵ E.

Note *a*. Since the plane of the polygon *a* . . . . *f* is horizontal, $\beta\ \gamma$ is equal and parallel to *b c*, and so on.

Note *b*. If the shadow of the *complete* pyramid had been required, it would have been necessary merely to determine the horizontal trace *t* of the ray through its vertex, and to join *t* B, *t* E.

A truncated right cone stands on the horizontal plane of projection. Determine its shadow upon that plane. Problem II.

Fig. 62, Pl. XVII. In Fig. 62, Pl. XVII., *h* is the horizontal trace of the ray through the vertex *v v'* of the cone. Then, if *q* and *p* are the points of contact of tangents drawn from *h* to the base of the cone, planes of rays touching the cone will cut the horizontal plane in *h q* and *h p*, while *v q*, *v' q'* and *v p*, *v' p'* will be the projections of their respective lines of contact with the cone. Hence *q a b p* is the shaded portion of the cone on plan, and *p' b'* is the line separating light and shade in elevation. If then a circle, with radius equal to that of the circular top of the truncated cone, is described having *s* (the horizontal trace of the ray through the centre of this circular top) as centre, this circle would touch *h q* and *h p*, and would complete the outline of the shadow cast by the truncated cone on the horizontal plane.

Note. *q h p* is the outline of the shadow which would be cast by the *entire* cone.

Problem III. A right cone has a horizontal axis inclined θ to the vertical plane. Determine its shadow on the horizontal plane.

Fig. 63, Pl. XVII. If a, β, γ, δ, Fig. 63, Pl. XVII., are the horizontal traces of rays passing through the extremities of the vertical and horizontal diameters of the circular base of the cone, and t is the horizontal trace of the ray through the vertex v v'; then, an ellipse with $a\,\beta$, $\gamma\,\delta$ as conjugate diameters being described, tangents to this ellipse from t complete the outline of the shadow cast by the cone on the horizontal plane. (See note, *p.* 40.)

To determine the lines of separation of light and shade on the cone, it is necessary to obtain, as in Prob. II., the lines of contact of planes of rays touching the cone. The plane of the base of the cone is vertical, and i is the point in which the plan of the ray through the vertex v intersects this plane. Hence i' (obtained by projecting from i on to the elevation of the ray through v') is the elevation of the intersection of this ray with the plane of the base of the cone. Therefore, if from i' tangents $i'\,p'$, $i'\,q'$ are drawn to the elliptic elevation of the base of the cone, these tangents will be the elevations of the intersections with the plane of the base of planes of rays touching the cone (see Theors. 29, *p.* 27; and 31, *p.* 38). Hence lines drawn from v' to p' and q' (the points of contact of the tangents from i') will be the elevations of the lines of contact of the planes of rays, and they are therefore the lines separating light and shade in elevation. The plans (p and q) of p' and q' are obtained by projection, and $v\,p$, $v\,q$ will be the lines of separation on plan.

Note *a*. The points of contact p', q' of the tangents from i' to the elliptic elevation of the base of the cone should be determined by one of the constructions of Plane Geometry.

The lines of separation of light and shade can be obtained in another way. If s is the point of contact of the tangent drawn from t to the ellipse $a \gamma \beta \delta$, then a ray through s will cut the plan of the base of the cone in p, and p' can be determined by projection; q and q' can be similarly obtained.

Problem IV.

A right cone (axis vertical) and a sphere rest on the horizontal plane. Determine the shadow cast by the cone on the sphere; the shadows cast by both solids on the planes of projection; and the shaded portions of the solids themselves.

Fig. 64, Pl. XVIII.

In Fig. 64, Pl. XVIII., the projections of both solids and of the ray of light are shewn. The required operations are as follows:

1st. Suppose planes of rays to touch the cone, and determine the projections of the lines of contact of these planes, their horizontal traces, and the projections of their intersections with the sphere.

2nd. Suppose a cylinder of rays enveloping the sphere, and determine the traces of this cylinder and the projections of its circle of contact with the sphere.

The necessary constructions are explained below.

The shadow cast by the cone on the horizontal plane and the shaded portions of the cone are first determined, as in Prob. II. of this Chapter. Lines drawn from $h\,t$ (the horizontal trace of the ray through the vertex) to touch the circular base of the cone in a and b determine the shadow of the cone on the horizontal plane and the shaded portion ($a\,v\,b$) of the cone *on plan*. The shaded portion of the cone *in elevation* is obtained by drawing lines from v' to a' and b', the elevations of a and b respectively.

A portion of the shadow of the cone does not fall on the horizontal plane, but is intercepted by the sphere. This portion will be now determined.

To determine the outline of the shadow cast by the cone on the sphere, it is necessary to cut the latter solid by planes of rays touching the former. These planes will have $h\,t$, a and $h\,t$, b, for their horizontal traces; they will touch the cone in the generatrices $v\,a$, $v\,b$; they will be both inclined at an angle α equal to the angle at the base of the cone (Theor. 29, *p.* 27), and they will cut the sphere in circles whose projections will be ellipses. If a ground line $z\,z$ be drawn at right angles to $h\,t$, b, cutting it in f and touching the circular plan of the sphere, this plan will then become an elevation of the sphere on $z\,z$, and $f\,e'$, making α with $z\,z$, will be the vertical trace (on the plane $z\,z$) of the plane of rays touching the cone in $v\,b$ (Theor. 26, *p.* 26). Thus $e'\,e'$ is the elevation on $z\,z$ of the circle in which the plane $e'f\,b$ cuts the sphere, and $e\,e$ (drawn through c, the plan of the centre of the sphere, parallel to $z\,z$ and terminated by perpendiculars dropped from e', e') is the minor axis of the ellipse into which this circle is projected on plan. The major axis of this ellipse is equal to $e'\,e'$, and bisects $e\,e$ in p (see *p.* 40). A portion of this ellipse is now drawn and its elevation on $x\,y$ deduced. By a precisely similar construction the section of the sphere by the other plane of rays which touches the cone in $v\,a$, and has $h\,t$, a for its horizontal trace, is determined. The projections of this section also are ellipses, and these ellipses must respectively cut the two ellipses first obtained in i and i', the projections of a point on the ray through the vertex of the cone. This affords a check on the previous construction.

It is now necessary to determine the shaded portion of the sphere and the shadow cast by it on the planes of projection.

The rays intercepted by a sphere form a right cylinder, and the outline of the shadow cast by the sphere on the planes of projection is that of the section of the cylinder by those planes. This outline is therefore made up of two ellipses, one of which is the vertical and the other the horizontal trace of the cylinder of intercepted rays. These two ellipses must intersect on $x\,y$, or the outline of the shadow would not be continuous. Further, half the sphere will be in shade, and since the great circle which separates the light and dark halves is the circle of contact of the cylinder and sphere, its plane is at right angles to the direction of the rays, and its projections are both ellipses.

An elevation of the cylinder of intercepted rays is made on a ground line W W taken parallel to the plans of the rays and *touching* the plan of the sphere. (Thus this plan becomes an elevation of the sphere on W W.) Draw $m''\,s'$, $m''\,s'$ touching the plan (now elevation) of the sphere, and making with W W an angle β equal to the true inclination of the rays of light. The construction (Prob. II., Chap. II.) for determining β is not shewn. Then $m''\,m''\;s'\,s'$ is the elevation of the cylinder on W W, and $c\,o'$ parallel to $m''\,s'$ is the elevation of its axis. Thus $m''\,m''$ is the elevation (on W W) of the circle of contact of the cylinder and sphere. Now lines parallel to the rays on plan and touching the plan of the sphere in n, n determine the ***plan*** of the cylinder. Hence $n\,n$ is the plan of the horizontal diameter of the great circle of contact, and is therefore the major axis of the required ellipse. By projecting from m'', m'' the minor axis ($m\,m$) is obtained. The ellipse is now drawn and the elevation on $x\,y$ is deduced (see *p.* 40.)

It remains to determine the intersection of the cylinder of intercepted rays with the planes of the projection.

The point o (obtained by projecting from o') is the horizontal trace of the axis of this cylinder, and is therefore the centre of the ellipse in which the latter cuts the horizontal plane. The major axis of this ellipse will coincide with the plan of the axis of the cylinder, and s

(obtained by projecting from s') is one of its extremities. The minor axis is of course equal to the diameter of the sphere. The ellipse thus determined cuts $x\,y$ in two points, and that portion of it which is above $x\,y$ need not be drawn. The intersection of the cylinder with the vertical plane is another ellipse, the axes of which can be determined in the same way as those of the last ellipse; or, since only a small portion of the new ellipse is required, points on this portion can be obtained by drawing rays through a portion of the circumference of the great circle whose projections have been determined, and finding their *vertical traces.*

In the figure, the outlines of both cast shadows and shaded portions are dotted where they would be invisible. The outlines of both shadows and shaded portions are not continued beyond the points where they cut. Thus the outline of the shadow of the cone on the horizontal plane is not continued beyond the point in which it meets the outline of the shadow of the sphere, and similarly with respect to the shadow cast by the cone on the sphere and the shaded portions of the latter solid.

The above use of the plan of a sphere as its elevation should be noticed. The ground lines of the two auxiliary planes $z\,z$ and W W could have been assumed in any position, provided that the former was at right angles to $h\,t$, b, and the latter parallel to the plans of the rays; but by taking these ground lines *touching* the plan of the sphere, the latter can be used as its own elevation, and a saving of construction is therefore effected. **Note.**

CONVENTIONAL DIRECTION OF THE RAYS.

In the preceding problems the direction of the rays has been assumed at pleasure, and differs in each case. It is, however, evidently important that in Engineering Drawings some convention should be adopted, so that the rays of light may have a known and invariable direction.

This conventional direction of the rays is taken to be parallel to one of the diagonals of a cube standing on the horizontal plane, and having one face parallel to the vertical plane. Thus, in Fig. 65, Pl. XIX., $a\,b$ is the plan, and $a'\,b'$ the elevation, of one of the diagonals of a cube, and it is to this diagonal that the conventional rays of light are supposed parallel. The projections of the latter will therefore invariably make 45° with the ground **Fig. 65, Pl. XIX.**

line. The *real inclination* of these rays, the construction for determining which is shewn in the figure, is 35° 16′ very nearly.

Note *a*. In the case of an object lying on the plane of the paper, the rays may, roughly speaking, be said to come over the left shoulder.

Note *b*. French draughtsmen usually assume the rays to be parallel to the *other* diagonal of the cube.

Two problems are added, in both of which the rays of light are assumed to have the conventional direction above explained.

Problem V. **Determine on plan the shadow cast into the concave surface of a hemisphere resting on the horizontal plane, and also the shaded part of its section by a vertical plane passing through the centre and parallel to the ground line.**

Fig. 66, Pl. XIX. In Fig. 66, Pl. XIX., m and n, the points in which the *plans* of rays touch the plan of the hemisphere, are evidently the commencing points of the required curve, since it is at these points that the hemispherical surface begins to be illumined. Further, the outline of the shadow required is traced by the intersections with the hemisphere of all rays passing through the semicircular edge $m\,s\,n$. Make a section of the hemisphere on $s\,p$ the diameter parallel to the plans of the rays. This section is constructed on $z\,z$ parallel to $s\,p$. Then since the rays of light are parallel to the section plane, their elevations will make with $z\,z$ an angle equal to their real inclination (Theor. 14, *p.* 19). Now a line from s' (the elevation of s), making 35° 16′ with $z\,z$, cuts the semicircular section of the hemisphere in t'. Hence t, obtained by projecting from t', is the plan of the point in which a ray through s strikes the hemispherical surface, and is therefore a point on the outline of the required shadow.

Similarly, the semicircle $a'\,\beta'\,d'$ is the common elevation on $z\,z$ of imaginary sections of the hemisphere on $a\,b$ and $c\,d$, parallel to $s\,p$. From a', the common elevation of a and c, draw $a'\,\beta'$ parallel to $s'\,t'$, then by projection β and γ are the plans of the intersections with the hemisphere of rays through a and c, and are therefore points on the outline of the required shadow. By taking more imaginary sections on lines parallel to $s\ p$, and proceeding in the same way, any number of points on the required curve are obtained.

Note. The angle $t'\ s'\ o'$ = the angle $\beta'\ a'\ o'$; but the angle $t'\ s'\ o'$ = half the angle $t'\ o'\ p'$, and $\beta'\ a'\ o'$ = half $\beta'\ o'\ d'$ (Euc. III., 20). Hence the angle $t'\ o'\ p'$ = the angle $\beta'\ o'\ p'$; or, in other words, the point β' is on the radius $t'\ o'$. Similarly, the elevation of any other point on the outline of the shadow cast into the hemisphere can be proved to lie on $t'\ o'$. Hence the real outline of this shadow is a plane curve, and all plane curves lying on a spherical surface are circles. Therefore, the curve $m\ \beta\ t\ \gamma\ n$ is an ellipse whose semi-axes are $m\ o$ and $o\ t$. Hence the points m, n, and t being obtained, the required outline of the shadow can be drawn at once.

It remains to determine the shaded part of the section of the hemisphere by a plane through o parallel to $x\ y$. This section (the semicircle $a''\ d''$) is shewn in Fig. 66, and is constructed on $x\ y$.

Draw the *elevation* of a ray touching the semicircular outline of the section of the hemisphere in e', and determine e the plan of e', and m'' the elevation of m. Then the portion of the outline of the section which casts the required shadow is $a''\ m''$, $a\ m$ on the horizontal, and $a''\ e'$, $a\ e$ on the vertical great circle. Determine β'' the elevation of β on $z\ y$; then β'' is a cusp on the required outline of the shadow. Further, the portion of outline of this shadow between β'' and m'' is obviously the elevation of the portion $m\ \beta$ of the previously determined ellipse $m\ t\ n$; this portion can therefore be obtained. To find points on the remaining portion of the outline between β'' and e', take a section of the hemisphere through q (any point between a and e) by a vertical plane parallel to the plans of the rays. The elevation of this section on $z\ z$ is the semicircle $h'\ r'$, and since q' on this semicircle is the elevation of q, the line $q'\ i'$ parallel to the *real direction* of the rays, and cutting the semicircle, determines i' the elevation of a point in which a ray through q cuts the hemispherical surface. By projection, i the plan of i' is obtained, and i'', the elevation of i on $x\ y$, is a point on the required outline of the shadow between β'' and e'. Similarly other points on the outline of this portion of the shadow can be obtained.

Note. The *true inclination* of the rays, which has been made use of in this problem, should be obtained by construction, as in Fig. 65.

Problem VI. **A cylindrical bolt having a cylindrical head stands on the horizontal plane, the axis being vertical. Determine the shadow cast on the shank by the projecting head.**

Fig. 67, Pl. XIX. An elevation and half plan of the bolt are shewn in Fig. 67, Pl. XIX. Then the plans of rays touch the plans of the head and shank in b and a respectively; generatrices at these points will, therefore, be the bounding lines of the unilluminated portions. Now it is the *lower* edge $c'\ d'$ of the head which casts a shadow, hence it is necessary to draw rays through this edge and determine their intersections with the shank of the bolt. Thus $f\ f'$, $h\ h'$ are the projections of two points on the lower edge of the head. Through f and h draw the plans, and through f' and h' draw the elevations of rays. The former cut the *plan* of the bolt in 1 and 2 respectively; hence 1′, 2′ (the elevations of 1 and 2 obtained by projection) are points on the required outline of the shadow. Similarly, other points can be determined.

Note *a*. The commencing point of the visible outline of the shadow will evidently be the point 3′ obtained by the intersection with the bolt of the ray e 3, e' 3′, which passes through the bounding generatrix 3 l' of the shank.

Note *b*. The curve 3′, 1′, 2′, &c., need not be drawn beyond the point where it meets the outline of the unilluminated portion of the bolt. The outline of the invisible portion of the shadow is not shewn.

SHADED PORTIONS OF SOLIDS OF REVOLUTION.

The outline of the shaded portion of a solid of revolution can always be determined in the following way:

Suppose the solid to be cut by a series of planes passing through its axis, obtain the projection of the ray on each of these planes, and *in* each plane draw a line tangent to the section of the solid and parallel to the projection of the ray on that plane. Then the points of contact of these tangents are points on the required outline of the unilluminated portion.

Thus, in the case of an annulus generated by a circle, and having a vertical axis (e. g. the "torus" round the capital of a column), any plane $z z$ passing through the axis will be vertical, and will cut the solid in circles whose plans are straight lines. Then the elevation (r'') of the ray on the plane $z z$ being first determined, the section of the annulus by that plane is rotated about the axis of the solid into a vertical plane parallel to $x y$. The *elevation* of this section will then be a circle, coinciding with the outline of the elevation of the annulus. Tangents to this circle, making with $x y$ the same angle that r'' makes with $z z$, must now be drawn, and the points of contact of these tangents must be rotated back into the original position of the section plane, thus determining points on the plan of the outline of the unilluminated portion of the solid. The elevations of the points thus obtained can be readily deduced. By making use of several section planes in the way above indicated, other points on the required outline can be determined.

Note. If tangents are drawn to the plan of any solid of revolution parallel to the plans of the rays, the points of contact of those tangents will be the extremities of the visible outline of the dark portion of this view of the solid, and similarly with respect to the elevation of the solid and of the rays.

EXAMPLES.

(N.B. In the following examples the rays of light are supposed conventional, unless otherwise specified.)

1. The vertex of an *inverted* right hexagonal pyramid (height 4", side of base 1·25") is in the horizontal plane. One edge is in the vertical plane, and the axis is vertical. Determine the dark portion of the pyramid and the shadow cast on *both* planes of projection.

2. A pentagonal right prism (side of base 1·5", height 4") has one edge of its base in the horizontal plane; the corresponding edge of its opposite end is in the vertical plane, and the axis is inclined at 50°.

Determine the dark portion and the shadow cast on *both* planes of projection. (The rays are inclined at 40°, and their plans make 45 with the plan of the axis of the prism.)

3. An octahedron (2·5″ edge) has its lowest face parallel to, and 1″ above the horizontal plane. The centre of this face is 2″ from the vertical plane, and one of its edges makes 30° with that plane. Determine the shadow cast on the planes of projection.

4. An inverted right cone (height 3·5″, radius of base 1·25″) has its vertex in the horizontal plane, and axis vertical. The base *touches* the vertical plane. Determine the dark portion in plan and elevation, and the shadow cast on the plane of projection.

5. A hollow inverted right cone (radius of base 1·5″, height 2·75″) has its axis vertical. It is truncated at a distance of ·75″ from the vertex by a plane parallel to its base. Determine the shadow cast on its *interior* surface.

6. The centre of a sphere of 1·25″ radius is 1·5″ and 2″ from the horizontal and vertical planes respectively. Determine its shaded portion in plan and elevation, and the shadow cast on both planes of projection.

7. Determine the shadow of the same sphere on a plane inclined 40°, and having a horizontal trace touching the plan of the sphere, and making 45° with $x\ y$.

8. An ellipse (axes 3″ and 2″) generates a spheroid by rotation about its major axis, which is *vertical.* Determine the dark portions of its projections and the shadow cast on the *horizontal* plane.

9. A vertical cylindrical bolt (diameter 2″) has a hexagonal head (side 1·5″). Determine the shadow cast by the head on the shank.

10. An annulus is generated by a circle of 2″ diameter whose centre moves in a circle of 3″ diameter. The axis of the annulus is vertical. Determine its shaded portion in plan and elevation, and the shadow cast on the horizontal plane.

11. A square-threaded screw (diameter 3″, pitch 1″) stands vertical. Determine the shaded portion of its elevation.

12. Determine the shaded portion of the section of the corresponding female screw by a vertical plane through the axis.

13. Determine the shaded portion in elevation of a vertical V-thread screw, diameter 3″, pitch 1″, angle of thread 55°.

Note. In the following examples it is intended that the data as to dimensions of the solids and direction of the ray should be assumed by the student or indicated by the instructor. The shadows cast on both the planes of projection should be determined in each case.

14. Determine the shadow cast by a vertical cone on a horizontal cylinder.

15. Two equal intersecting right octagonal prisms each have one edge of the base in the horizontal plane, the axes bisecting each other at right angles. Determine the shadow cast by one prism on the other, and by the compound solid on the horizontal plane.

16. Determine the shadow of a vertical pyramid on a vertical prism.

17. Determine the shadow of a vertical cone on a vertical prism.

18. Determine the shadow of a sphere on a vertical cone.

19. Determine the shadow of a sphere on a vertical cylinder.

20. Determine the shadow of a sphere on a horizontal prism.

21. Determine the shadow of a vertical cone on an anchor ring resting on the horizontal plane.

22. Determine the shaded portion of the vertical section of a vertical cylinder with a piston fitting into it. (The piston to be in elevation.)

23. Determine the shaded portion of a horizontal section of a cylindrical boiler with hemispherical ends.

CHAPTER X.

SHADE-LINES AND SHADING.

SHADE-LINES.

Shade-lines. SHADE-LINES are now very extensively employed in Engineering Drawings. When properly placed they render a single view of an object more intelligible by giving the appearance of relief to its parts.

In shade-lining a drawing the following rules should be strictly observed:

Rule 1. Shade-lines denote the intersection of two surfaces, one of which is illuminated, the other in shade, and *the latter invisible.* Shade-lines should therefore never be placed at the junction of two surfaces, both of which are visible up to their line of intersection.

Note. If the *visible* surface has a curved outline, the shade-line would begin at the points at which the projection of the ray touches this outline, and its full strength should be gradually gained, starting at those points.

Rule 2. Shade-lines should never be drawn to indicate the outline or "contour" of a curved surface.

Note. Thus the elevation of a vertical right cylinder would be a rectangle, and the only shade-line in this elevation would be the line representing the base of the cylinder. On the other hand, the elevation of a vertical square prism, with one face parallel to the vertical plane, would be a rectangle also, but one of the sides as well as the base would be a shade-line.

Rule 3. In shade-lining a drawing, the rays of light are supposed to take the conventional direction explained in page 107. Thus, if a plan and *one* elevation of an object are to be shade-lined, the rays in the former will come from the *left-hand bottom corner* of the paper, and in the latter from the *left-hand top corner*, making 45° with the edges of the paper in each case. The direction of the rays in the case of other elevations will be understood by referring to Fig. 68, Pl. XIX. In this figure, the plan of a cube resting on the horizontal plane is shewn, and also the elevations of its four vertical faces. Then in the elevation of the— Fig. 68, Pl. XIX.

Front face, a c, that drawn on *x y*; the ray proceeds from the *left* top corner.

Left-side face, a d, that drawn on 22; the ray proceeds from the *right* top corner.

Right-side face, b c, that drawn on 44; the ray proceeds from the *left* top corner.

Rear face, d b, that drawn on 33; the ray proceeds from the *right* top corner.

The above directions of the rays are of course those which would be apparent to a spectator facing the respective planes on which the elevations are made. Note *a*.

The four elevations might have been *all* constructed on *x y*, but the direction of the ray with respect to each elevation would be unaltered. Note *b*.

In the figure, each of the four vertical faces of the cube is distinguished by a projection or a recess. Thus the face *a c* has a circular recess, *a d* a square recess, *b c* a square projection, and *d b* a circular projection. These projections and recesses are correctly shade-lined in their respective elevations. Note *c*.

It is better never to use shade-lines to a drawing on which cast shadows or shading are employed. **Rule 4.**

Always place shade-lines so that their thickness is *outside* the outline of the object. **Rule 5.**

Shade-lines to *sections* follow the above rules, the direction of the ray in any particular section depending upon the position of the cutting plane. **Shade-lines to sections.**

In Fig. 69, Pl. XIX., a side elevation of the end joint of the lower boom of a girder bridge is given, shewing the two cast-iron saddles by means of which an articulated bearing is secured, and also the rollers supporting the lower saddle and provided for the expansion and contraction of the girder. This figure well exemplifies the value of shade-lines, and should be carefully studied. Fig. 69, Pl. XIX.

Note. It will be noticed that shade-lines are shewn half round the rivet-heads, which would strictly indicate that the latter were of cylindrical instead of spherical form. This treatment of rivets is, however, a recognised convention, and one from which no mistake would be likely to arise.

SHADING.

Shading is employed to render the *true form* of an object intelligible in a single view. Thus, a vertical cylinder in elevation requires a plan in order that its *true form* may be recognised, while a *shaded elevation* of the same cylinder explains itself.

Shading in Engineering Drawings is but little used, on account of the time and labour it requires: a knowledge of its rules is, however, very desirable, since a few properly disposed shades can frequently be made to add considerably to the clearness and general effect of the view of an object.

The following rules should be strictly observed:—

Rule 1. *Illuminated plane surfaces parallel to the plane of projection* receive flat tints. The nearer the surface is to the eye, the lighter such tints should be.

Rule 2. *Illuminated plane surfaces inclined to the plane of projection* receive graduated tints, becoming *darker* as such surfaces recede from the eye. Of two such surfaces unequally inclined, that which receives the rays most directly is lightest.

Rule 3. *Unilluminated plane surfaces parallel to the plane of projection* receive flat tints. The nearer the surface is to the eye, the darker such tints should be.

Rule 4. *Unilluminated plane surfaces inclined to the plane of projection* receive graduated tints, becoming *lighter* as the surface recedes from the eye.

Note. The tint of the darkest portion of an illuminated surface should be approximately equal in depth to that of the lightest portion of an unilluminated surface.

For shading, two brushes (which may conveniently be placed at opposite ends of the same holder) are required, one for laying on the tint, the other for softening off its edges with clean water. The two brushes should be of good size, and blotting paper should be at hand for emptying either at pleasure. The shading tints should be laid on in narrow strips, beginning always with the *darkest* portions of the object. Before the edges of these strips have time to dry, they should be softened off with the clean brush, which should be only just moistened for the purpose. When the first strip is quite dry, the next should be laid over and beyond it, and softened off in the same way. It is better not to try to attain the required depth of shade too quickly. The greater the number of tints, the softer the general appearance, and the more delicate the gradation will be.

Shading by strips.

Shading by strips alone, without softening off, is sometimes practised, especially by French draughtsmen. Bold and striking effects are obtained by this method.

Figs. 70, and 71, Pl. XIX.

In Figs. 70 and 71, Pl. XIX., the elevations of two equal vertical hexagonal prisms are shewn, and shaded respectively by strips and by softened washes.

Curved surfaces.

Fig. 72, Pl. XIX.

Curved surfaces are shaded in accordance with the above rules; they require, however, some further explanation. In Fig. 72, Pl. XIX., the elevation and half plan of a vertical right cylinder is shewn. A line parallel to the plans of the rays touches the plan of the base of the cylinder in d, hence the portion of the cylinder from $a a'$ to $d' d'$ in elevation is illuminated, the remaining portion ($d' d'$ to $e e'$) being in shade. Now the light falls on this cylinder most directly at the generatrix l, $l' l'$; this should therefore (Rule 2) be the lightest portion. But from $l' l'$ to $o' o'$ succeeding generatrices approach the eye, the surface about these generatrices ought, therefore, to be continually lighter (Rule 2). In order to comply with both the above considerations, bisect the angle $l o' o$ by $o' r$, and make the portion $l\, l'\, r'\, r'$ the lightest portion in elevation. This portion need, in fact, receive no shading tint, and may remain white. Again, the darkest portion of the cylinder is evidently about the generatrix *d′ d′*. Thus, from *r′ r′* to *d d′* the shade will increase in depth, and from *d′ d′* to *e e′* it will diminish.

Figs. 72 and 73, Pl. XIX. The cylinders shewn in Figs. 72 and 73 are shaded in accordance with the above considerations; the former by strips, the latter by softened washes.

The cylinder has been selected as an example of shading on account of its very frequent recurrence in machine drawing. Other curved surfaces, as the cone and sphere, require merely the intelligent application of the principles above laid down.

Cast shadows. Cast shadows, i. e. shadows cast by one portion of an object upon another, are treated according to Rules 3 and 4; their darkest portions are those on which the rays, if not intercepted, would fall most directly.

Topographical Drawing. In Topographical Drawing shading is employed independently of any supposed condition with regard to the rays of light. In such drawings a shading tint merely signifies *inclination to the plane of projection.* The greater this inclination, the darker the shade is made. It is on this principle that shaded plans of hill districts are tinted.

CHAPTER XI.

ISOMETRIC PROJECTION.

ISOMETRIC PROJECTION is seldom used in Engineering Drawings. It is, however, sometimes valuable in explaining small complicated details, and a knowledge of its principles affords some aid in making rough explanatory sketches.

If three straight lines, meeting in a point and mutually at right angles, are *equally inclined* to the horizontal plane, their plans will make equal angles with each other; three equal segments of the original lines will be represented in plan by three lengths equal to each other; and generally, any distances set off anywhere on the three original lines, or on any lines parallel to them, will have lengths in plan bearing always a fixed relation to the respective original distances. **Theorem 36.**

For example, imagine a cube resting with one angular point on the horizontal plane, the diagonal through that point being vertical. Then the three edges of the cube meeting in either extremity of the vertical diagonal are equally inclined to the horizontal plane; and the planes of the three faces bounded by these edges also make equal angles with the horizontal plane. Hence, the plans of these three edges will be three lines making 120° with each other, and the outline of the complete projection of the cube will be a hexagon. This bounding hexagon, together with the plans of the edges which meet in the extremities of the vertical diagonal, constitutes the "isometric projection," or "isometric view," of the cube.

It will be evident, therefore, that isometric projection is simply the orthographic projection of an object specially situated with respect to the planes of projection, this special position being such that all lines parallel to the length, breadth, or height of the object make equal angles with the horizontal plane.

Isometric axes. Isometric lines.

The three imaginary lines parallel to which in an isometric view the length, breadth, and height of the various portions of an object are taken, are called "isometric axes," and any lines parallel to these axes are termed "isometric lines." The isometric axes, therefore, always make 120° with each other, while all right angles contained by isometric lines are represented by angles of either 60° or 120°.

Fig. 74, Pl. XX.

In Fig. 74, Pl. XX., the three lines *o x, o y, o z,* meeting in *o* and making 120° with each other, are the isometric axes; equal lengths, *o a, o b, o c* are set off on each, and from *a, b,* and *c,* lines are drawn parallel to the other two axes, determining the points *d, e,* and *f.* The resulting figure *a d b e c f* is evidently a hexagon, and is the isometric projection of the cube before referred to. It will be recognised as the orthographic plan of a cube, one diagonal of which is vertical. Thus *a o, b o, c o,* are the plans of the three edges meeting in the upper extremity of this diagonal, while *d o, e o, f o* would be the plans of the three *invisible* edges meeting in its lower extremity.

Advantage of isometric projection.

Thus, by the simplest possible construction, a projection of the cube is obtained, in which three faces (*a d b o, b e c o, c f a o*) are visible. This then is the advantage of isometric projection: it affords a perfectly simple method of obtaining an oblique view of an object in such a position that several faces are simultaneously presented to the eye. Thus an isometric view can be made to do duty for a plan and two orthographic elevations of an object. At the same time it is evident that isometric projection is only adapted to objects in which all or most of the principal lines are mutually at right angles.

Fig. 74, Pl. XX.

In the figure, the points *a* and *b* being equally distant from the horizontal plane, the diagonal *a b* is horizontal. Hence *a b* is the length of the real diagonal of a face of the cube. If a square is constructed on *a b* as diagonal, its side will be equal to the real length of the edge of the cube; $a\mathrm{D}b$ is half this square, and if r is the middle point of the diagonal ab, then :—

$$ar = ad \cos. \angle\, dar = a\mathrm{D} \cos. \angle\, \mathrm{D}ar;$$

but $$\angle\, dar = 30°, \text{ and } \angle\, \mathrm{D}ar = 45°.$$

Therefore, $$ad . \frac{\sqrt{3}}{2} = a\mathrm{D} . \frac{1}{\sqrt{2}}\ ; \text{ or}$$

$$ad : a\mathrm{D} = \sqrt{2} : \sqrt{3}.$$

This, then, is the invariable relation between the isometric projections and *true lengths* of lines.

Isometric scale. The *inclination* of the isometric axes is cos. $\sqrt{\frac{2}{3}} = 35°$. Thus a scale may be constructed for isometric drawings, the divisions of which bear to those of the true scale of the drawing the ratio $\sqrt{2} : \sqrt{3}$, and a scale so constructed will give the length to be set off on any isometric line to represent any given real dimension. In practice, however, an isometric scale is never used, dimensions taken off the *true scale* being set off along the isometric lines. Provided that this is remembered, no mistake can arise from representing lines isometrically longer than they really should be, since all lines are increased in the same ratio.

Fig. 74, Pl. XX. In Fig. 74, an elevation of the isometrically situated cube is drawn on a vertical plane gg parallel to four of its edges, and perpendicular to two of its faces. The lowest extremity of the vertical diagonal is supposed to be in the horizontal plane.

Note. On the face $cobe$ of the isometric cube (Fig. 74, Pl. XX.) a square block is shewn projecting from, and symmetrically situated with respect to that face. The height st of this block has been made equal to one third of the edge of the cube. Similarly on the face $fcoa$ of the same cube a square recess is shewn, having a depth equal to one fourth of the edge of the cube.

Isometric projection of circles.

The isometric projection of a circle will of course be an ellipse (see *p.* 40), and the latter can always be obtained by determining the isometric projections of any two diameters of the original circle which are at right angles to each other, and constructing the ellipse on these lines as conjugate diameters. If, however, the isometric projection of a small arc of a circle only is required, it will be simpler to draw the chord and a few ordinates of the arc, and then place the chord and ordinates in isometric projection. In Fig. 74, Pl. XX., the lines $p\,p$, $q\,q$, parallel to $o\,x$ and $o\,y$ respectively, are the isometric projections of two diameters at right angles to each other of the circle inscribed in face $o\,a\,d\,b$ of the cube; an ellipse described with $p\,p$ and $q\,q$ as conjugate diameters will therefore be the isometric projection of that circle.

Again, $m\,m$ is the diameter of the circle described in the square a D b, and is therefore the plan of the *horizontal diameter* of the circle described in the face $o\,a\,d\,b$ of the cube. Hence $m\,m$ is the *major axis* of the elliptic plan of this circle (Note, *p.* 40). The minor axis $n\,n$, of this elliptic plan can be obtained by making $n'\,n'$ in elevation equal to $m\,m$ and projecting. But $m\,m$ is equal to the real edge of the cube, and—

$$\frac{n\,n}{m\,m} = \frac{n\,r}{m\,r} = \frac{r\,o}{r\,a} = \tan.\,30^\circ = \frac{1}{\sqrt{3}}.$$

This, therefore, is the ratio between the axes of an isometric ellipse.

Fig. 75, Pl. XX.

Fig. 75, Pl. XX., shews an isometric view of the top or capital of one of the columns supporting the roof of a railway station. Two cast-iron facia girders serve to connect the tops of adjacent columns, and are bolted right through those tops, as shewn. The top of the column has a hollow projecting bracket, which supports a cast-iron bowl (not shewn in the figure); the gutters (also of cast-iron) hook into this bowl, and discharge their rain-water into it. The latter passes down through the bowl and the hollow bracket into the interior of the column, finally entering a drain laid parallel to the line of columns. On the back portion of the top of the column rests a cast-iron bed-plate, and this bed-plate and the wrought-iron brackets, to which the extremity of the T-iron principal rafter is riveted, are bolted down on to the top of the column by two $\frac{7}{8}''$ bolts. The holes for these bolts are shewn in the figure, but the cast-iron bed-plate has been removed.

The drawing of this isometric view presents no difficulty, but requires care, on account of the many different planes which contain the various surfaces.

The curves with which the lower flanges terminate are obtained from their elevations by ordinates.

The scale shewn is the true scale of the drawing, and has been used for the isometric view.

CHAPTER XII.

PERSPECTIVE.

It is evident that the methods of orthographic and isometric projection do not represent objects as they really appear. The further an object is from the eye, the smaller it seems to be, but in orthographic projections the distance does not affect the question. If an elevation of a house on any plane is drawn, then the elevation on all parallel planes will be precisely the same, whatever distance the latter may be from the position in which the house is supposed to stand.

If, however, a pane of glass were placed between the eye and any object, and all the lines of the latter were traced out on the glass exactly in the positions in which they appear to the eye, then a view is obtained termed a "perspective" from the idea of "looking through" a transparent plane.

It is clear that the eye, the plane, and the object must occupy fixed positions with respect to each other. That is to say, that while the operation of tracing out the outline of the object on the glass is being performed, neither the eye, plane, or object must move relatively to each other, or the picture would be hopelessly confused. For example, if during the operation the glass were moved nearer to the eye, the picture traced on it would become smaller, and the two portions traced at the two different positions would not fit. It is also evident that only one eye must be supposed to be used during the process, otherwise each eye would have its own perspective, and the two views, unless the object were very distant, would interlace and confuse each other. The eye must in fact be considered as a point.

Note.

This is easily tested. Place a dot on a window pane. Shut the right eye, and with the left get the dot in line with some point outside the window. Now, without moving the head, shut the left eye and open the right. Unless the point selected is comparatively distant, it will be seen that the dot no longer appears in line with it. If the outside point is distant the displacement still takes place, but is so small as not to be apparent.

Definition 1. "Perspective."

If a transparent plane is placed between the eye and any object, and on it the outlines of all portions of the object are traced exactly as they appear to the eye, then the view traced on the transparent plane is called the "perspective projection," or simply the "perspective" of the object, and all methods of obtaining that view are summed up under the general term "Perspective." The above can be stated in a more geometrical way as follows. If from every point of an object lines are drawn, all of which converge in a fixed point—the eye—these lines cutting any plane interposed between the eye and the object will trace out a figure termed the "perspective" of the object.

Definition 2. "Visual rays."

The lines converging in the eye may be regarded as rays of light issuing from the object, and are called "visual rays."

Definition 2. "Picture Plane."

The interposed plane is called the "picture plane," or "picture."

Definition 3. "Station Point."

The fixed point, or the eye, is called the "station point."

Definition 4. "Ground Plane."

It is evident that the perspective of an object cannot be obtained unless the relative positions of the eye, the plane, and the object are known; and this can best be effected by referring them, as in orthographic projection, to an imaginary horizontal plane termed the "ground plane."

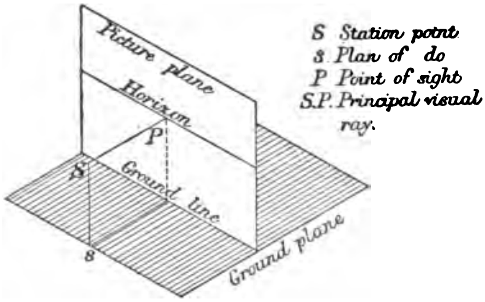
Definition 5. "Ground Line."

The picture plane is, in all practical perspective drawing, taken as vertical, and the line in which the picture plane cuts the ground plane—the horizontal trace of the picture plane (see p. 25)—is called the "ground line."

The above definitions are illustrated in Fig. 76.

It follows from these definitions that;—(1) the perspective of a point is always a point: (2) the perspective of a straight line is always a straight line, except in the case in which it passes through the station point, when it becomes a point: (3) the perspective of a plane figure is always a plane figure, with the same number of sides, except in one case. If the plane of the figure passes through the station point, its perspective becomes a straight line. (4) The perspective of a plane curve is always a curve, except when the plane containing it passes through the station point, in which case the perspective becomes a straight line.

Fig. 76.



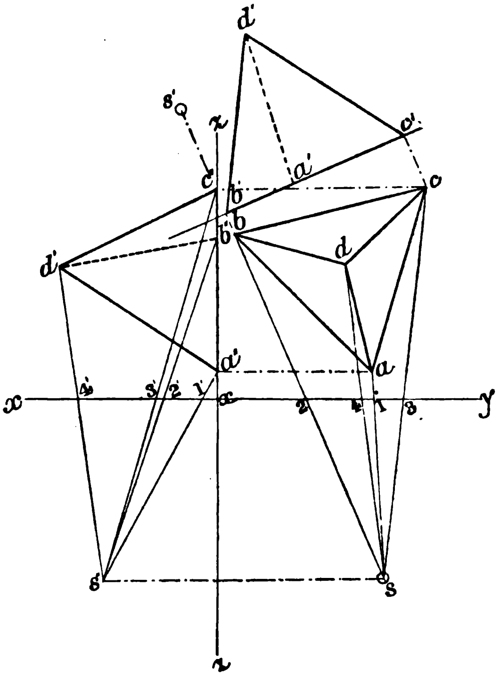
PERSPECTIVE BY SIMPLE ORTHOGRAPHIC PROJECTION.

Tetrahedron. Fig. 77.

Perspective of a Tetrahedron.—The operation involved in perspective is really that of determining the intersection of a number of convergent lines with a plane. This operation can be performed by the ordinary methods of orthographic projection as follows: $a\,b\,c\,d$, $a'\,b'\,c'\,d'$ (Fig. 77) are the projections of a tetrahedron, $s\,s'$ those of the station point. It is required to obtain the perspective of the tetrahedron on a picture plane of which $x\,y$ is the ground line. Draw $s\,a$, $s\,b$, $s\,c$, $s\,d$. These lines, cutting $x\,y$ in 1, 2, 3, 4, will be the plans of visual rays proceeding from S to the angles of the solid, and the points 1, 2, 3, 4, will be the plans of the required perspectives of A, B, C, D. It is only necessary to obtain the heights of these perspectives above $x\,y$. Take a new ground line $z\,z$ anywhere at right angles to $x\,y$, and make a new elevation $a''\,b''\,c''\,d''$, of the given solid on $z\,z$. Similarly, obtain s'' the elevation of S on $z\,z$

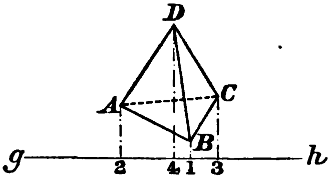
and draw $s''a''$, $s''b''$, $s''c''$, $s''d''$, which will be the elevations on zz of the four visual rays. Then the points $1', 2'$, $3', 4'$, in which these elevations cut xy produced, evidently

Fig. 77.



give the heights of the perspectives required. To construct the perspective it will be more convenient to transfer the ground line xy to gh to avoid confusing the plan. Transfer

Fig. 77A.



the points 2, 4, 1, 3, with a slip of paper to gh (Fig. 77A), and erect a perpendicular from each. Then set off on the perpendiculars the respective heights of $1', 2', 3', 4'$, above *z z*, and the figure A B C D is obtained, which is the perspective of the tetrahedron under the given conditions.

The student is recommended to practise several similar cases, choosing positions for the picture plane and station point. Figs. 39 and 40, Pl. IX; 45, Pl. XI; 47, Pl. XII., will serve for the purpose, and can be traced or pricked off.

The above method would prove in many cases too tedious. In drawing the perspective of a house, for example, the number of lines would become very great and the figure would be liable to be confused. If, however, it were required to obtain the perspective of a geometrical figure given by its projections, as in the cases above referred to, this method would probably prove the best. Practical perspective has to deal almost exclusively with architectural subjects, and in their case a different mode of procedure must be adopted.

PERSPECTIVE BY VANISHING POINTS.

Use of Vanishing Points.

It will be noticed that in the method above adopted the perspective of a line is obtained by determining the perspectives of two points on it. In fact the whole picture has to be built up point by point, and it is this which makes the process usually too tedious for practice.

General Principles.

The following principles and definitions, which are peculiar to perspective proper, and do not enter into the operations of orthographic projection, should be thoroughly mastered.

If any system of parallel lines running away from the eye be studied, as for instance in looking along a straight length of railway line, it will be seen that all the lines seem to converge towards a point. This is necessarily the case, since the distance apart of the lines must seem less and less as the lines themselves recede from the eye.

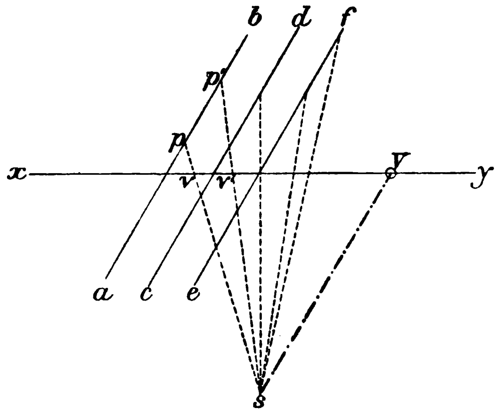
Definition 6. "Vanishing Point."

The point in which any system of parallel lines appears to converge is termed the "vanishing point" of that system.

Fig. 78.

Let *x y*, Fig. 78, be the ground line of the picture plane, *ab* any line lying in the ground plane, *s* the station point, also in the ground plane. Take any point *p* on *ab*, and join *sp*, cutting *xy* in *v*. Then *v* is the perspective of *p*. Suppose *p* to move along *ab* to *p'*. Join *sp'*, cutting *xy* in *v'*. Then *v'* is the perspective of *p'*. If *p* is now supposed to move further along *ab*, its perspective will move to the right along *xy*, and finally if *p* moves to an infinite distance along *ab*, then the line *sp* will become parallel to *ab*. Draw *s V* parallel to *ab*, cutting *xy* in V. Then V will be the perspective of a point on *ab* at an infinite distance from *s*. But if any other lines *cd*, *ef*, be drawn parallel to *ab*, precisely the same reasoning can be applied to them, and it can be argued that the perspective of a point on either of them at an infinite distance from *s* is *V*. Thus we may say that *V* is the vanishing point of *ab* and of all lines in the ground plane parallel to it. But similar reasoning will apply to any system of parallel lines whatever. Hence, generally, the vanishing point of any line or system of parallel lines is found by drawing a visual ray parallel to the line (or lines) to cut the picture plane.

Fig. 78.



Several important deductions may be drawn from the above.

Lines parallel to the picture plane have no vanishing point, and their perspectives are parallel lines. In other words, their vanishing point is at an infinite distance. Thus the vertical lines of an object are represented by vertical lines in its perspective, since the picture plane is usually taken as vertical. If the picture plane were inclined forwards, vertical lines would have a vanishing point below the level of the station point; if backwards, above. Thus it is that in some photographs a convergence of the vertical lines of an architectural subject may be noticed. The sensitive plate was not vertical when the picture was taken, and the latter is therefore a perspective on an inclined plane.

Definitions 7, 8. "Principal visual ray." "Point of sight."

The vanishing point of all lines perpendicular to the picture plane is the foot of the perpendicular dropped from the station point on that plane. This perpendicular is called the "principal visual ray," and its foot the "point of sight" (Fig. 76). The point of sight is thus the vanishing point of all lines perpendicular to the picture, and is that point on the picture which is exactly opposite the eye, or at which the eye is supposed to look fixedly. The horizontal line drawn in the picture through the point of sight is called the "horizon" (Fig. 76). Thus the horizon is a horizontal line in the picture at the same level as the eye.

Definition 9. "Horizon."

It follows that the horizon must contain the vanishing points of all horizontal lines, except lines parallel to the picture, which have no vanishing point.

Definition 10. "Vanishing Line."

By similar reasoning it can be shewn that all systems of parallel planes have "vanishing lines" except when they are parallel to the picture, and that the horizon is the vanishing line of all horizontal planes. A vertical line through the point of sight is the vanishing line of all vertical planes perpendicular to the picture. The vanishing line of every system of parallel planes can always be found by determining the intersection with the picture of a plane parallel to the system and passing through the station point. Further, the vanishing line of a plane contains the vanishing points of all lines in that plane or parallel to it. Thus the vanishing line of any plane can be found by joining the vanishing points of any two lines in the plane, or parallel to it, but not themselves parallel. In practical perspective, the conception of the vanishing line of a plane is, however, rarely required.

It will be evident that the use of vanishing points is of great service in shortening the operations of perspective. Having obtained the vanishing point of any line, it is necessary to obtain only one point on any second line parallel to it. The second line can then be drawn through this point to the vanishing point of the first line, and its position is fixed, and similarly for any number of parallel lines. In an architectural subject, in which there are usually a great number of parallel lines, the operations of perspective are thus much assisted.

VANISHING POINT—HORIZONTAL LINES.

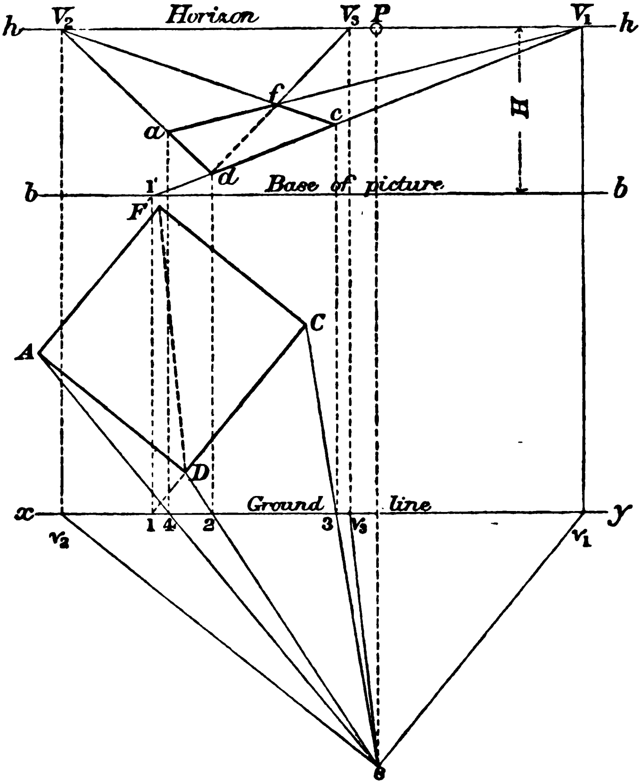
Horizontal Lines. Fig. 79.

In Fig. 79, A D C F is a square lying in the ground plane; *s* the plan of the station point, the height of which is H above the ground; *x y* the ground line. Take any line *b b* parallel to and at a convenient distance from *x y*. This line, which is merely the ground line transferred in order to avoid confusing the perspective view with the plan, is called the "base of the picture." Draw *h h* parallel to *b b*, and a distance *H* from it. Then *h h* is the horizon, and *P*, obtained by drawing a perpendicular from *s* on *x y* and projecting up, is the point of sight. Draw $s\, v_1$, parallel to A F, or D C, and cutting *x y* in v_1. Then v_1 is the *plan* of the vanishing point of A F and D C. But these lines being horizontal must have their vanishing points in the horizon. Project up from v_1 on to *h h*, obtaining V_1 which is the vanishing point of A F and D C. Similarly V_2 is the vanishing point of D A and C F and V_3 is the vanishing point of the diagonal D F. Supposing that the square were lifted up to any height above the ground plane, remaining horizontal, the construction for obtaining the vanishing points would be exactly the same. This construction should be clearly understood,

Definition 11. "Base of picture."

as the vanishing points of horizontal lines are always required at the outset of a perspective drawing, and it is

FIG. 79.



possible to dispense altogether with the vanishing points of lines oblique to the ground plane.

VANISHING POINTS OF OBLIQUE LINES.

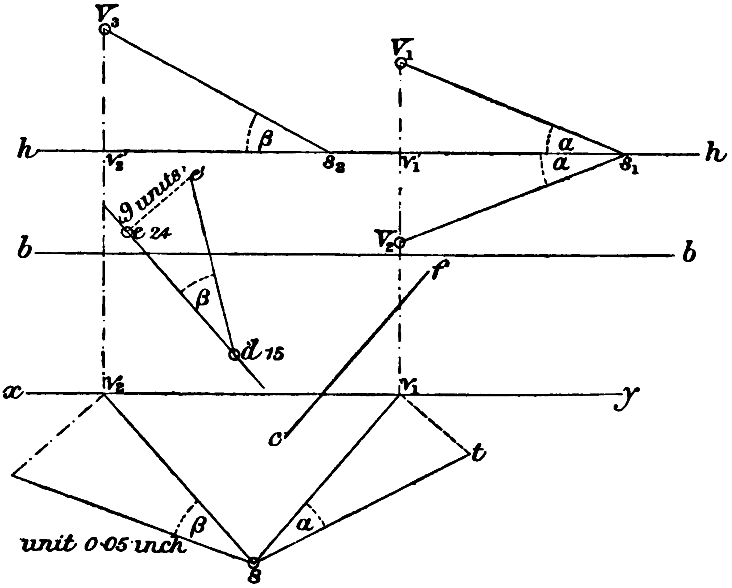
Oblique Lines.

Fig. 80.

In Fig. 80, *cf* is the plan of a line inclined at *a* to the ground plane. It is required to determine the vanishing point of this line. As before, *s* is the plan of the station point, *b b* the assumed base of the picture, *h h* the horizon.

Draw $s\,v_1$ parallel to *c f*, cutting *x y* in v_1, and project up from v_1 on to the horizon *h h*, obtaining v_1'. Then v_1' would be the vanishing point of the line *c f* if it were

FIG. 80.



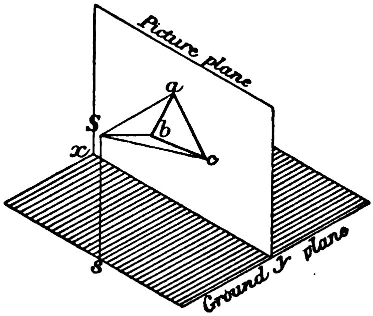
horizontal, and evidently the point in which a line through the station point parallel to *c f* cuts the picture plane—that is the vanishing point of *c f*—must be vertically above or below v_1', according as *f* is higher than *c*, or *vice versâ*. On $s\,v_1$ as base construct a right-angled triangle $s\,v_1\,t$, having the angle $v_1\,s\,t$ equal to *a*, the given inclination of *c f*. Then $v_1\,t$ is the height above or below v_1' of the required vanishing point. Make $v_1'\,V_1$ equal to $v_1\,t$, then V_1 is the vanishing point of *c f* if *f* is higher than *c*, or in other words if the line *c f* rises as it recedes from the eye. Similarly, making $v_1'\,V_2$ equal to $v_1\,t$, V_2 is the vanishing point of *c f* if *c* is higher than *f* or if the line falls as it recedes from the eye. Or, the right-angled triangle may be constructed on *h h*, making $v_1'\,s_1$ equal to

$v_1 s$ and the angle $v_1' s_1 V_1$ equal to α. On the same figure, the vanishing point V_3 of a line $d\,e$ given by two figured points is determined. The inclination β of this line is first found by the right-angled triangle $d\,e\,e'$, and the remaining construction is then the same as before. (Note *c*, p. 45.)

All lines *in* the picture plane are supposed to have their real length, and no diminution of size consequent on the distance of the eye from the picture is taken into account. Fig. 81. Thus if *a b c*, Fig 81, is a triangle in the plane of the picture, it is its own perspective wherever the eye may be placed.

Since any point *in* the picture is its own perspective, it follows that if any line cuts, or is produced to cut, the picture plane, the point of intersection is a point on the perspective of that line.

FIG. 81.



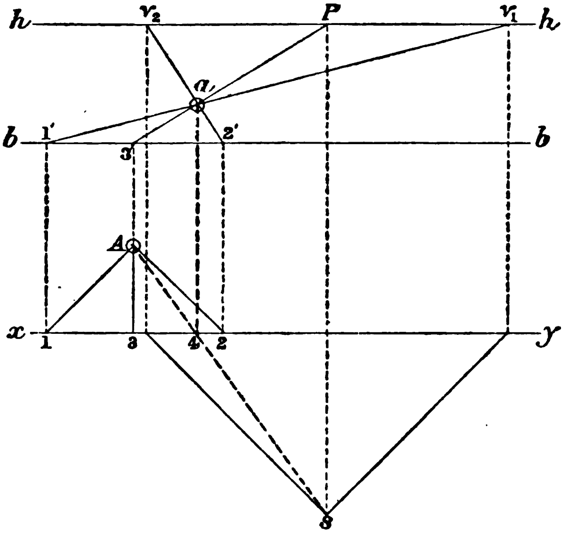
A POINT IN THE GROUND PLANE.

Perspective of a point. Fig. 82. In Fig. 82, *A* is a point on the ground plane of which the perspective is required. The station point, base of picture, and horizon are assumed, and lettered as before.

1st method. 1. Draw any two lines through *A*, cutting *x y* in 1 and 2. Assuming these to be lines in the ground plane, determine their vanishing points v_1 v_2. Transfer the points 1 and 2 to the base of the picture by projecting up. Join

$1'v_1$, $2'v_2$. Then $1'v_1$ is the perspective of the line $1A$ produced indefinitely, and $2'v_2$ is the perspective of $2A$. Hence a, their intersection, is the required perspective of A.

FIG. 82.



2. Or, draw any line $1A$ through A, and obtain its vanishing point v_1 as before. Draw a second line $A3$ from A at right angles to xy. Then $3A$ may be looked upon as a line in the ground plane perpendicular to the picture. Thus P, the point of sight, is its vanishing point. Project 3 up to the base of the picture, and join $3'P$, cutting $1'v_1$ in a. 2nd method.

3. Again, draw any line $2A$ through A as before, and find its perspective $2'v_2$. Join As, cutting xy in 4. Then As is the *plan* of the visual ray from s to A, and 4 is the plan of the perspective of A. Hence projecting up from 4 on to the line $2'v_2$, a is obtained. 3rd method.

It will be seen from the above that there are usually several ways of dealing with a perspective question, and only practice can in some cases decide which is the best to adopt. If a point is not well determined (Rule 6, p. 4) by one method, another may be employed to check it.

A Plane Figure in the Ground Plane.

Perspective of a square. Fig. 79.

A F C D, Fig. 79, is a square lying in the ground plane. The other conditions are assumed.

Determine V_1, V_2 the vanishing points of the pairs of parallel sides of the square. Then, to start the perspective view, produce one side C D to cut *x y* in the point 1; project 1 up to the base of the picture and join 1′ V_1. Now obtain on this line the perspectives of D and C by joining D *s*, C *s*, and projecting up from the points 2, 3 where these lines cut *x y*. Then *d* and *c* are the perspectives of D and C. From *d* and *c* draw lines to V_2. Joining *s A*, and projecting up from 4, *a* is determined. Join *a* V_1 cutting *c* V_2 in *f*. The perspective is now complete.

The figure *a f c d* might have been obtained in several other ways. Thus only one vanishing point V_1 might have been found, and the line 1′ V_1 with the points *d*, *c* on it obtained as before. Then producing F A to cut *x y*, projecting up from the point of intersection on to *b b*, and joining the projected point to V_1, a line is obtained on which the perspectives *a f* of *A* and *F* could be determined.

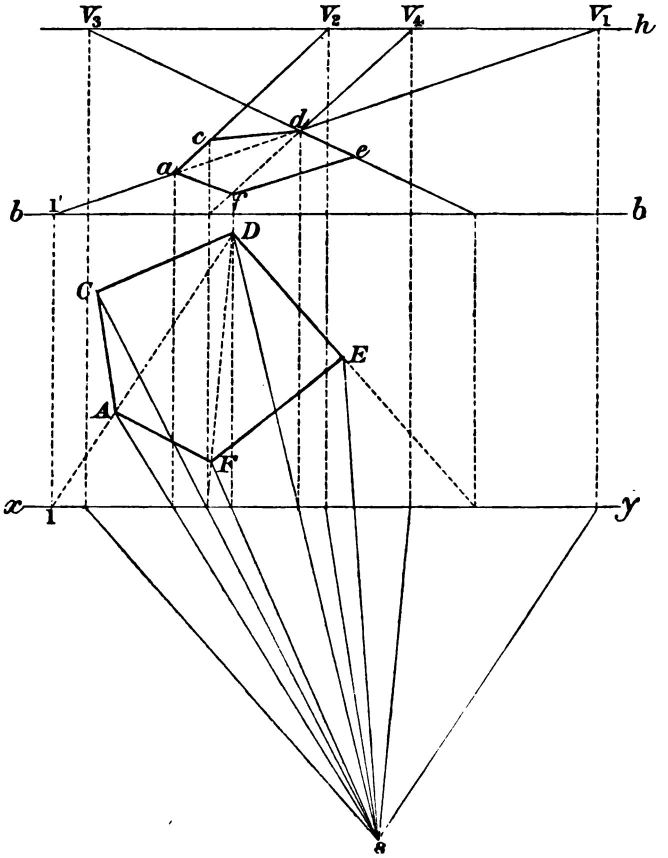
Perspective of any irregular plane figure. Fig. 83.

A C D E F, Fig. 83, is any irregular five-sided figure lying in the ground plane. This figure might be treated precisely like the square shewn in Fig. 79, but that the vanishing points of some of the sides—those of *A F* and *C D* for example—would be at an inconvenient distance.

Produce the diagonal *D A* to cut *x y* in 1, and obtain V_1 the vanishing point of *A D*. Project up from 1, and join 1′ V_1. Obtain *a*, *d* on this line by drawing the plans of the visual rays *s A*, *s D*, and projecting up as usual. Obtain V_2 the vanishing point of *A C*. Join *a* V_2 and determine *c* upon it. Similarly obtain V_3 the vanishing point of *E D*. Join V_3 *d*, and, producing it, determine *e*. Obtain V_4 the vanishing point of the diagonal *F D*. Join

V_4 d, and, producing it, obtain f. The perspective is now complete.

Fig. 83.



Although four vanishing points have been found above, the perspective could have been obtained with one only. Thus draw any series of parallel lines through the five points A, C, D, E, F to cut $x\,y$, and obtain their vanishing points. Project up from the points of intersection of these lines with $x\,y$ on to $b\,b$, and join all the projected points with the vanishing point. Thus five lines are obtained which severally contain the perspectives of the points A, C, D, E, F. Determine these perspectives by visual rays and projection. The assumed series of lines might be taken at right angles to $x\ y$, in which case the point of sight is their common vanishing point.

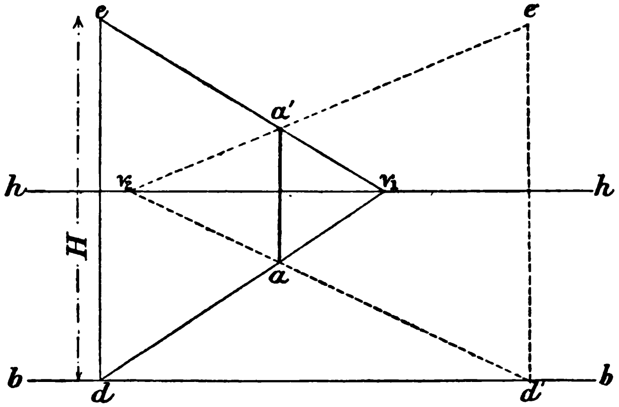
The employment of the intersection of the plans of the visual rays with $x\ y$ should be thoroughly understood, as it is a process which has to be carried out over and over again in perspective drawing.

POINTS NOT IN THE GROUND PLANE. LINES AND PLANE FIGURES PARALLEL TO THE GROUND PLANE.

Points not in the ground plane.
Fig. 84.

In Fig. 84, a is the perspective of the plan of a point whose height above the ground plane is equal to H. It is required to obtain the perspective of this point. Or, putting the problem another way, it is required to erect a perpendicular at a of height H. The horizon $h\ h$ and base of picture $b\ b$ are assumed. Draw any line through a to cut $h\ h$ in v_1 and $b\ b$ in d. At d erect a perpendicular $d\ e$ of length H. Join $e\ v_1$ and draw a vertical line up from a cutting $e\ v_1$ in a'. Then a' is the perspective required, or $a\ a'$ is the perspective of a perpendicular of height H erected at a point a in the ground plane.

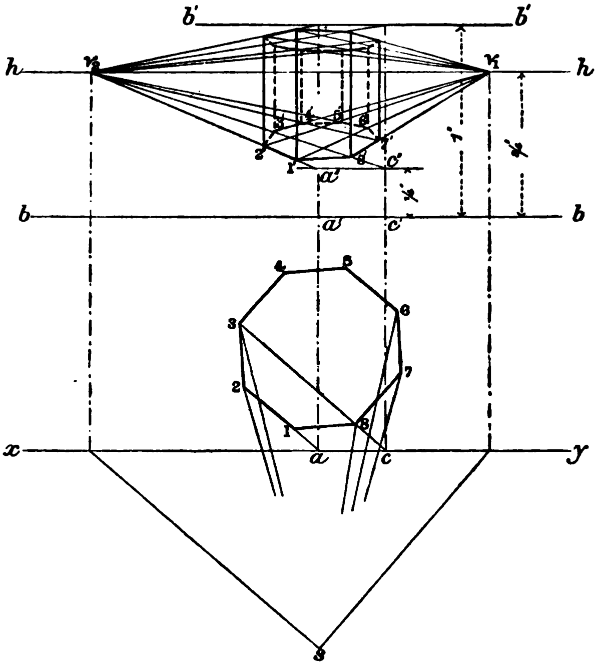
FIG. 84.



The reason of this construction is evident. Since $d\ v_1$ and $e\ v_1$ are lines which have v_1 a point in the horizon as vanishing point, they are the perspectives of parallel and horizontal lines, and therefore $a\,a'$ is parallel and equal (in perspective) to *d e*. But *d e* is in the picture and was made equal to H. Hence the height of *a* is *H*. It makes no difference where the line $d\,v_1$ is drawn. Thus in Fig. 84, the construction given by the lines $d'\,v_2$, $d'\,e'$, $e'\,v_2$ leads to the same result.

Since the perspectives of the plans of any required points can always be found by the construction previously given for points in the ground plane, the perspective of *any* point whose position is known can now be obtained.

FIG. 85.



Octagon parallel to ground plane. Fig. 85.

In Fig. 85, 1.....8 is the plan of an octagon whose plane is parallel to and at a distance of $\frac{1}{4}''$ above the ground plane. It is required to obtain its perspective, the ground line *x y*, the plan of the station point *s*, and its height above the ground plane (¾″) being given. Assume *b b* the base of the picture anywhere parallel to *x y*. Parallel to *b b* and a distance from it, equal to ¾″, draw *h h* the horizon. Determine v_1 the common vanishing point of the sides 8 7, 3 4 and the diagonals 1 6, 2 5, of the octagon. Similarly obtain v_2 the vanishing point of the sides 1 2, 6 5 and the diagonals 8 3, 7 4. Produce the side 2 1 to cut *x y* in *a*, and projecting upwards, set off a' a'' above *b b* equal to ¼″. Join a'' v_2, and on this line determine the perspectives 1′ 2′ of 1 and 2 by means of the plans of the visual rays in the usual way. Join 1′ v_1, 2′ v_1 and on the former line obtain 6′ the perspective of 6. Join 6′ v_2 which will cut 2′ v_1 in 5′ the perspective of 5. Produce 3 8 to cut *x y* in *c*, and projecting up make c' c'' equal to ½″. Join c'' v_2 and obtain on it 8′, 3′ the perspectives of 8 and 3. Join 8′ v_1, 3′ v_1, and on the latter obtain 7′ the perspective of 7. Join 7′ v_2 cutting 3′ v_1 in 4′ the perspective of 4. The figure is now complete.

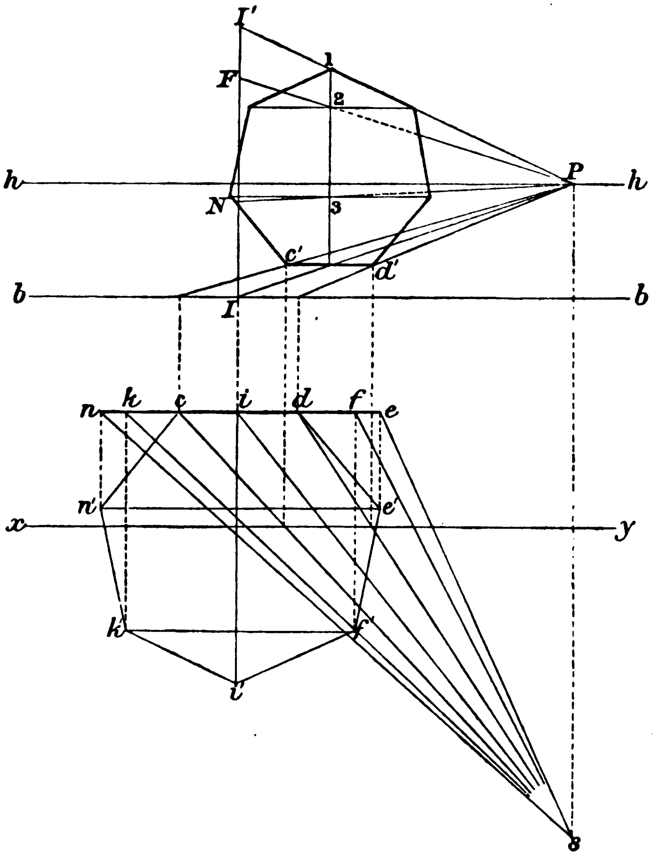
Vertical octagonal prism.

By repeating the above construction the figure can be extended so as to give the perspective of a vertical octagonal prism of any height having 1′....8′ as base. Thus, if the height of the prism were ¾″, it will only be necessary to produce the height lines a' a'', c' c'', setting off ¾″ on each of them above a'' and c'' respectively. Or the perspective of the top of the prism can be obtained in a different way. Draw b' b' parallel to *b b* and at a height above it equal to 1″, the height of the top of the prism above the ground plane. Produce the sides 2 1, 5 6, 7 8, 4 3, and the diagonals 3 8, 4 7, 6 1, 5 2 to cut *x y*, and project up all the points of intersection on to b' b'. Join the projected points to v_1 and v_2 according to which point the produced lines vanish in. Thus eight lines are obtained, four converging in v_1 and four in v_2. The intersections of these lines give all the points of the perspective of the top of the prism. This construction might of course have been employed in determining the perspective of the base 1′....8′.

PLANE FIGURE: VERTICAL AND PARALLEL TO THE PICTURE PLANE.

Heptagon parallel to the picture. Fig. 86.

FIG. 86.



In Fig 86, *n e* is the plan of a heptagon of which one side *c d* is in the ground plane. The plans *n k i f* of the angles of the heptagon are obtained in the usual way by turning the figure down about *c d*. Obtain P the point of sight. Project *c* and *d* up to *b b*. Join the projected points to P, and on the lines thus obtained determine c', d' the perspectives of c, d. Project up from i and use the line I I′ as a height line on which to set off I I′ equal to $i\,i'$, I F equal to $f\,f'$, and I N equal to $n\,n'$. Join I, I′, F, N to P, cutting a vertical line bisecting $c'\,d'$ in 1, 2, 3. Through the points 2, 3 draw lines parallel to $b\,b$, and on these lines obtain the perspectives of k, f and n, e respectively. The figure is now complete.

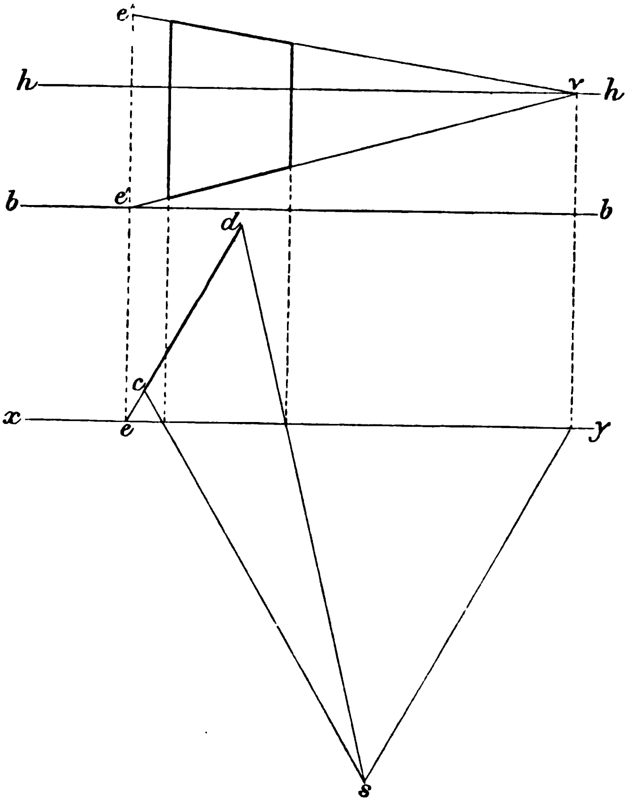
It should be noticed that the perspective of the heptagon is symmetrical about its centre line. This is only the case when its plane is parallel to the picture plane. If remaining vertical, it were moved so as to make an angle with the picture plane, the more distant half of the two horizontal diagonals and of the base would be foreshortened, and both these diagonals and the base would then have a vanishing point on the horizon instead of being parallel to $b\,b$.

PLANE FIGURE: VERTICAL AND INCLINED TO THE PICTURE PLANE.

Square. Fig. 87. (1.) In Fig. 87, $c\,d$ is the plan of a square of which the plane is vertical and one side is in the ground plane. Obtain v, the vanishing point of the two horizontal sides. Produce $d\,c$ to cut $x\,y$ in e, and projecting up make $e'\,e''$ equal to $c\,d$. Thus $e'\,e''$ is the height line for the plane of the square. Join $e'\,v$, $e''\,v$ and determine on both these lines the perspectives of c and d as shewn. The two vertical sides of the square are vertical in perspective, and the figure takes, therefore, the form of a trapezium.

Heptagon. Fig. 88. (2.) Fig. 88 shews the case of a vertical heptagon with one side in the ground plane. The construction is precisely the same as for the square. The plan must first be completed, as in Fig. 86, by turning down the heptagon. The vanishing point v_1 of $c\,d$ and of the two diagonals parallel to it, is then found. Producing $d\,c$ to cut $x\,y$ and projecting up, a height line $e'\,e''$ is obtained, on which are set off the heights of the two horizontal diagonals and of the top point of the heptagon. The several points on the height line are joined to v_1, and the perspectives of all the points of the hexagon are respectively determined on these lines.

FIG. 87.

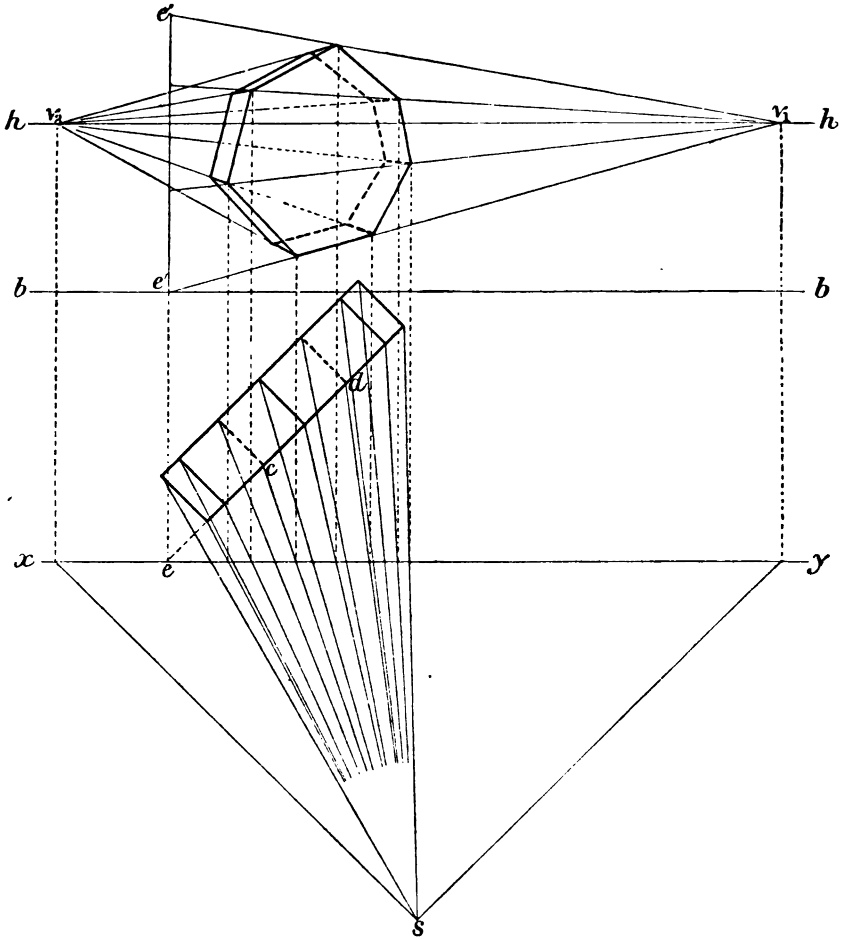


The figure can easily be extended so as to shew a heptagonal prism with vertical ends. Complete the plan of the prism, giving it any assumed length. Determine v_2 the vanishing point of the seven parallel edges of the prism, and draw lines to v_2 from all the points of the

Heptagonal Prism. Fig. 88.

already obtained perspective of the heptagon. From the plan determine on these lines the perspectives of the

Fig. 88.



points of the parallel heptagon which forms the further end of the prism.

Note. The perspective of the further end could be obtained without the use of the second vanishing point v_2 by producing the plan of this end to cut $x\,y$, and working with a new height line.

The employment of the height line in the two examples given above should be noticed. Every vertical plane not parallel to the picture plane will have a height line which can be used to determine the height of any point in it. Thus in dealing with such a subject as the side of a house, which might have a considerable amount of detail, the height line should if possible be found. It will also be convenient, if many heights have to be transferred to the height line, to draw an elevation of the object (or enough of it to give all heights) on the base of the picture produced. The height of any point can then be found on any height line by using the T-square.

Use of the Vanishing Point of Oblique Lines.

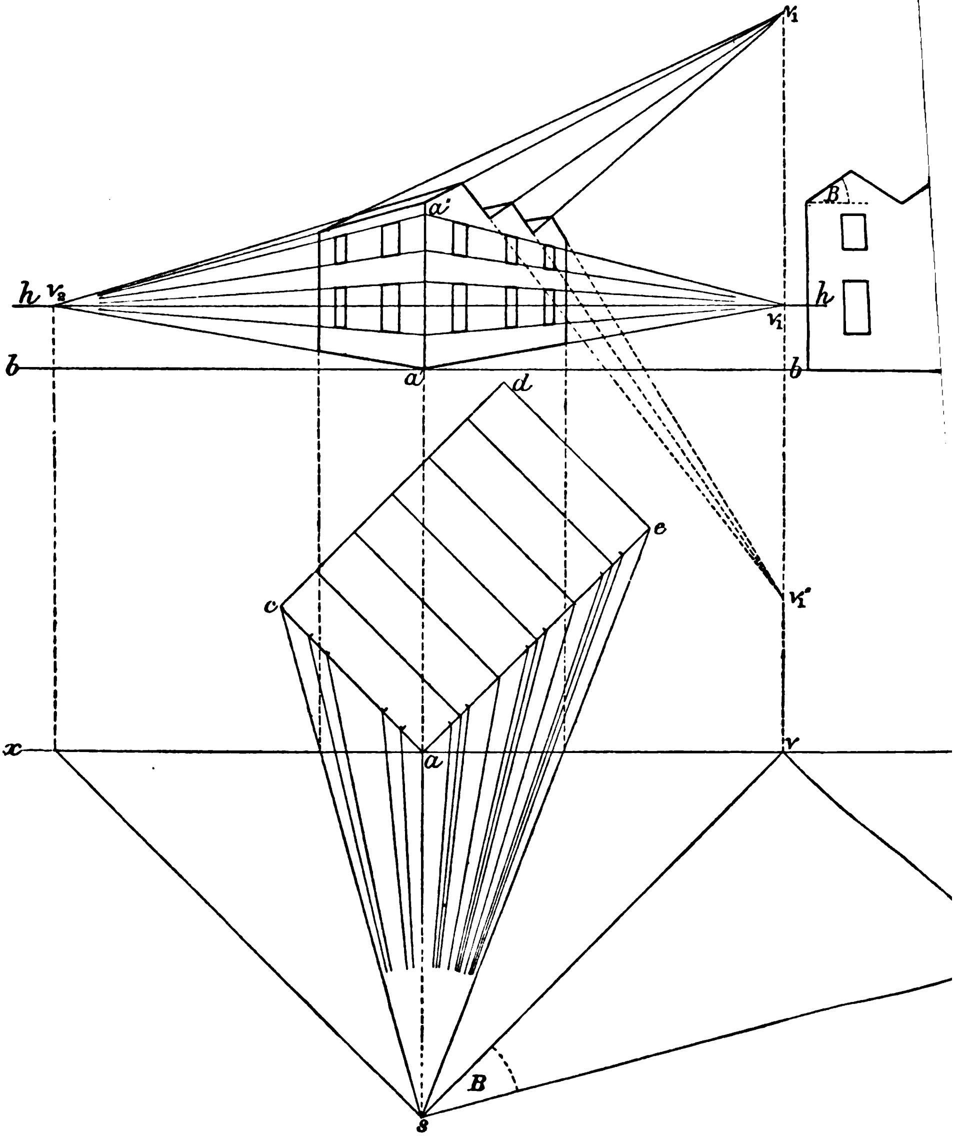
Gable-ended building. Fig. 89.

In Fig. 89, *a c d e* is the plan of a building covered by three ridge roofs gable-ended at *a e* and *c d*. A portion of the end elevation is drawn on *b b*, the base of the picture produced. The vanishing points v_1, v_2 of the ends and sides of the building are first obtained. Making the angle *v s t* equal to *B*, the pitch of the roof, set off $v_1 v'_1$ *above* v_1, and $v_1 v''_1$ *below* v_1, both equal to *v t*. Then v'_1, v''_1 are the vanishing points for the gable ends. Since one angle *a* of the building is *in* the picture plane, a vertical line projected up from *a* will serve as a height line for both the end *a e* and the side *a d*. Make *a′ a″* equal to the height of the eaves, and join $a'' v_1$. On this line determine the ends of valleys from the plan, as shewn. Then, drawing lines alternately to v'_1, and v''_1, the gables are put in. The rest of the figure needs no explanation.

2nd method. Fig. 90.

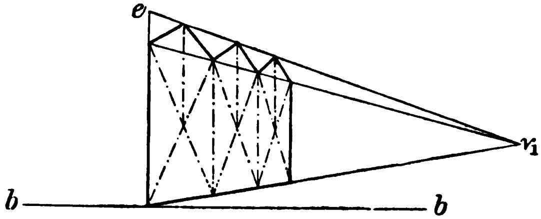
It is to be observed that the vanishing points v'_1, v''_1 can easily be dispensed with. Thus, setting off the height of the ridges on the height line, a point is obtained, which if joined to v_1 gives a line on which all the ends of the ridges can be obtained. Or, draw diagonals under each gable, as shewn in Fig. 90. Then vertical lines

FIG. 89.



drawn through the intersections of these diagonals to cut a line drawn from *e* (on the height line) to v_1, will give the

Fig. 90.



ends of the ridges. In practice, it will generally be simplest to work with only two vanishing points, and these both in the horizon, unless the number of parallel oblique lines is very considerable.

Circles.

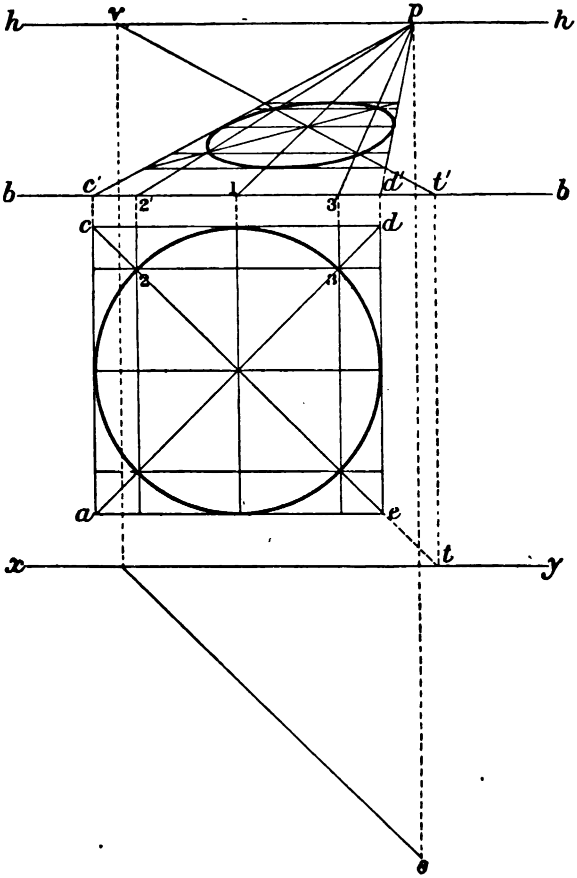
Circles.

The perspective of a circle is a straight line if its plane passes through the station point. If its plane is parallel to the picture plane and its centre at the same height as the eye, the perspective is a circle. In all other cases which occur in practice it is an ellipse. In the general case, the circle should be circumscribed by a square, and the perspective of the square obtained. If, further, the perspectives of the four points of contact and of the four points in which the circle cuts the diagonals of the square are determined, the ellipse can then be drawn with sufficient accuracy.

Circle in ground plane. Fig. 91.

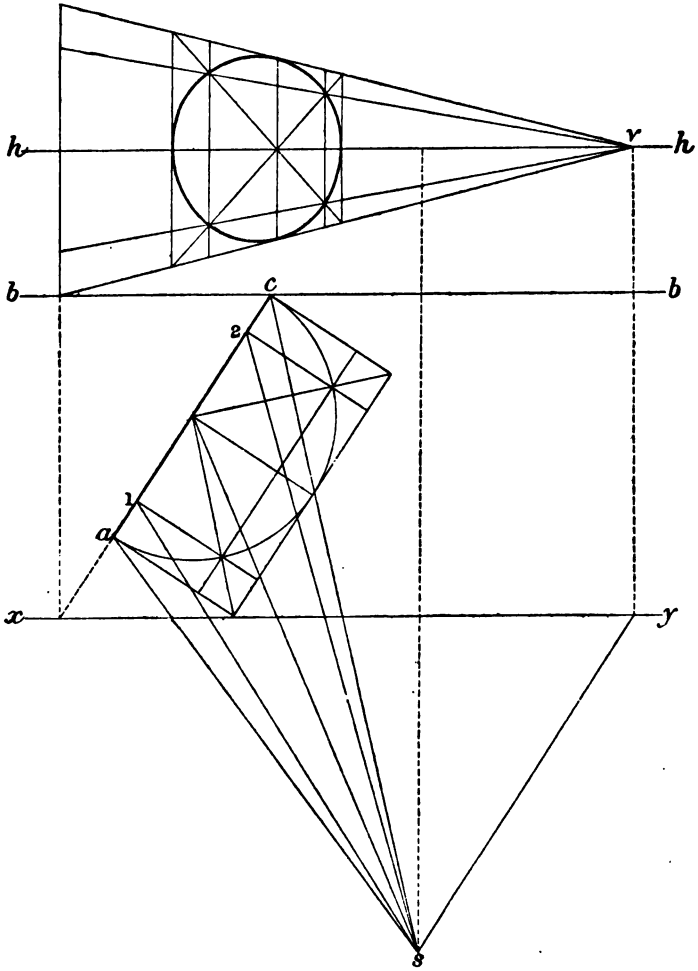
In Fig. 91, the given circle, lying on the ground plane, is circumscribed by a square *a c d e*, whose sides are (for simplicity) taken parallel and perpendicular to *x y*. Two sides of this square will, therefore, vanish in *p*, the point of sight, and the others will be parallel to *b b*, the base of the picture. Produce *a c*, *e d* to cut *b b* in *c'*, *d'*: join the points *c'*, *d'* to *p*. Obtain *v* the vanishing point of the diagonal *e c*. Produce *c e* to cut *x y* in *t*, and projecting *t* up to *b b*, join *t' v*. Lines parallel to *b b* drawn through the points where *t' v* cuts *d'p*, *c'p*, complete the perspective of the square. Draw a line through the centre of the circle perpendicular to *b b*, and join the point 1 where it cuts *b b* to *p*. Then 1 *p*, cutting the two parallel sides of the perspective of the square, gives two points of contact of the ellipse. A line parallel to *b b* drawn through the intersection of the diagonals of the perspective of the square gives the other two points of contact. On plan, draw the diagonals of the square cutting the circle in the points 2, 3. Project up to the points 2 and

FIG. 91.



3 on to *b b*, and join 2′ *p*, 3′ *p*. These lines cut the perspectives of the diagonals in four points on the ellipse. The latter can now be drawn.

Fig. 92.



Any number of additional points on the ellipse could of course be obtained, but four points, four tangents, and their points of contact will usually be sufficient.

Note *a*. The line 1 *p* evidently bisects the perspectives of the two sides of the square parallel to *x y*.

Note *b*. The perspective of the square obtained is a trapezium, and an ellipse inscribed in this trapezium by the methods of plane geometry would be the required perspective of the circle.

Vertical circle. Fig. 92.

The case of the vertical circle, Fig. 92, hardly requires explanation. The semicircle and circumscribing rectangle are drawn on *a c*, the plan of the given circle, in order to obtain 1, 2, the plans of the points in which this circle cuts the diagonals of the circumscribing square. The rest of the construction is shewn.

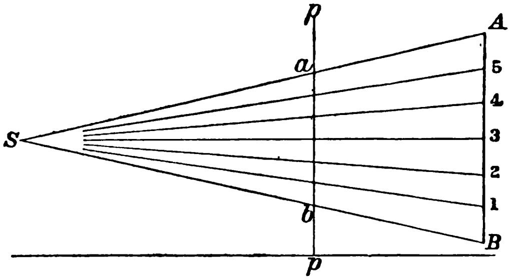
Division of Lines.

Lines parallel to picture. Fig. 93.

Vertical lines and all lines parallel to the picture plane can be divided up into any number of equal parts, or in any given proportion, in perspective, just as if the lines themselves were being dealt with.

In Fig. 93, *pp* is an *end view* of the picture plane, *S* the station point, *A B* a vertical line. Then *a b* is the perspective of the line *A B*, and if *A B* is divided up into any number of parts (six), the points of division joined to *S* will give lines which will divide up *a b* into *equal* parts. This reasoning can be applied to any line parallel to the picture plane.

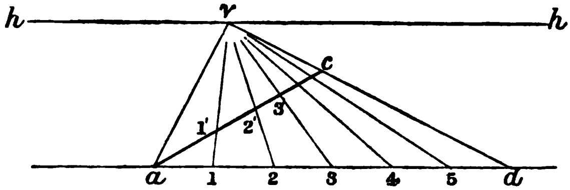
Fig. 93.



Horizontal lines. Fig. 94.

The case of a horizontal line not parallel to the picture plane is different. In Fig. 94, *a c* is the perspective of a horizontal line in any position, *h h* being the horizon: it is required to divide *a c* into six equal parts. Through *a* draw a line *a d* parallel to *h h*. Take any point *v* on *h h*. Join *v a*, *v c*, and produce *v c* to cut the line *a d* in *d*.

FIG. 94.

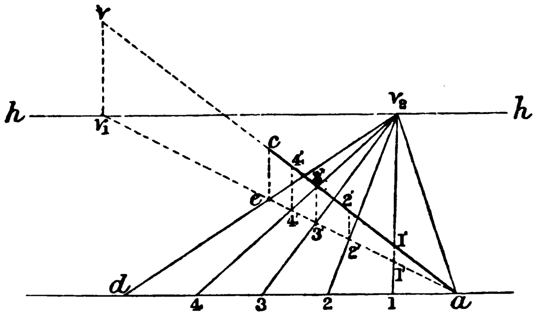


Divide *a d* into six equal parts, and through the points of division 1, 2, ... draw lines to *v* cutting *a c* in 1′, 2′, ... These lines will divide *a c* perspectively into six equal parts. For the lines which converge in *v* must be the perspectives of a set of parallel horizontal lines cutting *a c*. Hence *a* 1′ is equal to 1′ 2′, and so on. Similarly, if it were required to divide *a c* in any given proportion, it would be necessary merely to divide *a d* in that proportion, and through the points of division to draw lines to *v*.

The perspective of a line inclined to the ground plane can be dealt with if its vanishing point is known. Thus

Oblique lines. Fig. 95.

FIG. 95.



a c (Fig. 95) is the perspective of a line whose vanishing point is *v*, *h h* being the horizon: it is required to divide *a c* into five equal parts From *v* drop a perpendicular

$v v_1$ on $h h$ and join $v_1 a$. From c draw a line perpendicular to $h h$, cutting $v_1 a$ in e. Divide $a e$ into five parts by the construction of the previous paragraph, and through 1′, 2′ . . . the points of division, draw lines parallel to $c e$ or at right angles to $h h$, cutting $a c$ in 1″, 2″. . . . . These lines will divide $a c$ perspectively into five equal parts. For $a e$ is evidently the perspective of a horizontal line vertically below $a c$. Hence if $a e$ is divided in any given way, vertical lines through the points of division must divide $a c$ in the same way.

The three constructions above given may occasionally save trouble, as they enable the perspective of a line to be divided up in any required way without reference to the plan.

EFFECT OF VARYING THE POSITION OF THE STATION POINT.

Effect of moving the station point.

The size of the perspective of an object will greatly depend on the relative positions selected for the eye and the picture with respect to it. Thus, supposing the object and the picture plane to be fixed, the perspective will be increased in size by moving the station point away from the latter. On the other hand, supposing the station point and the picture plane to be fixed, the perspective will diminish in size as the object is moved further from the latter.

Again, the form of the perspective of any object will be much altered by raising or lowering the height of the station point, other conditions remaining the same.

It is not necessary that the object should be behind the picture plane. A portion, or the whole, may be in front of the latter. The construction will be precisely the same, but the perspective of any portion of the object which is in front of the picture plane will be greater than that portion itself.

Finally, it is desirable that the object should nowhere subtend a greater angle at the station point than 60° *at the outside*, or 30° on each side of the principal visual ray.

If a larger angle is included, more of the object is taken in than the eye can properly compass, and the result is that the perspective appears distorted and unnatural.

Note.

The limits of this angle—termed the "visual angle"—have been variously given, and doubtless vary in different individuals. It may be taken, however, that 60° is a maximum for an architectural perspective, and a less angle is desirable.

The distortion above referred to is sometimes to be noticed in photographs of architectural subjects taken with a lens which has too wide an angle.

PERSPECTIVE OF A HOUSE (FIGS. 96, 97, 98, Pl. XXI.)

A house. Figs. 96, 97, 98, Pl. XXI.

From what has been said above, it will be evident that the selection of the position of the station point is an important matter in dealing with architectural subjects.

In the case of a house, for example, the following points must be attended to:—

1. The building should not subtend a much greater angle than 40° at the station point.

2. The station point and picture plane should be so taken that two sides of the building are shewn. Prominence being given to the most architecturally ornamental side, a reasonable amount of the adjacent side should be brought into view. If there is too much of the latter, attention is withdrawn from the most important side; if too little, the features of the less visible side will be obscured and lost.

3. The height of the station point must be chosen with reference to the site. If the latter is high, a low station point may be taken; if low, a high station point. If the site is level, the station point should be taken at the natural level of the eye of a person standing—about 5 feet.

4. It will save trouble to make the ground line pass through the nearest angle of the plan of building. The angle of the latter can then be used as a height line for both the sides.

Arrangement of drawing.

The general arrangement of a perspective drawing is shewn in Pl. XXI. The plan and elevations would have been previously drawn in making the design. A tracing of the plan may be attached to the paper on which the perspective is to be drawn. The position of the station point is now assumed in accordance with the considerations above stated, and its plan is laid down on the paper. The ground line and the base of the picture should then be drawn. It will be convenient to arrange that the plan of the station point and the vanishing point for the sides of the building shall come on to the same drawing board or drawing table to which the paper is attached. This is not, however, absolutely essential, since an instrument called a centrolinead may be used to draw lines converging in an inaccessible point. It will be simpler, however, to use the actual vanishing points.

On the base of the picture as ground line, tracings of the two elevations (or as much of them as is required to give the heights of all points which will enter into the perspective) should be arranged. These tracings can be conveniently placed right and left of the place which the perspective will occupy, as shewn in Figs. 97, 98, Pl. XXI. They are not absolutely essential, since the heights can be transferred from the drawing by the dividers; but it will tend to prevent confusion and to simplify the work if they are provided. The two vanishing points V_1, V_2 of all horizontal lines in the right and left side of the house respectively are now determined in the usual way, and a pin is stuck into the drawing board at each of them. A third pin may also be used at the plan of the station point.

The perspective can now be begun, and will consist merely of the same kind of operation many times repeated. The angle of the house will serve as a height line for all lines lying in the plane of the two sides which meet in it. For other planes, such as that of the front of the porch and of the bow window, fresh height lines can be obtained. The process will consist mainly in:—1st, marking off heights on the height lines by projecting across with the

T-square from the elevations (Figs. 97, 98), and drawing lines to the respective vanishing points; 2nd, drawing lines from the plans of the various points to the plan of the station point, marking their intersections with $x\,y$, and projecting up. The latter operation would be done with the T-square, which would first be laid with its working edge against the pin, and passing through the plan of the point under consideration. The line need not be drawn, but its intersection on $x\,y$ would be marked, and then, by applying the T-square to the bottom edge of the board (removing the pin if necessary), the intersection is projected up. Care should be taken to put in no more lines than are absolutely essential, or confusion may result. If the number of intersections on $x\,y$ becomes very great, the existing ones may be rubbed out. Each portion of the building as it is completed should be put in in firm lines, all construction lines being kept as light as possible. In Fig. 96, Pl. XXI. the top plan only is shewn. The plans of other portions of the building—as for example the windows and doors—will be required in the course of the work, and could be taken on to the tracing from the first floor and second floor plans. It will not be necessary to treat small detail by construction, and a good draughtsman, having obtained all the principal lines in perspective, will be able to supply much by eye. But this can only be learnt from practice. The finishing up of the drawing, whether by pen and ink work as in Pl. XXI., or by colour, is also a matter on which it would probably be useless to attempt to give directions. Figures of persons should generally be introduced to give the requisite idea of size, and care should be taken that they are given, approximately, the heights which they would seem to have in the position in which they are placed.

PERSPECTIVE BY MEASURING POINTS.

Method of measuring points.

There is another method of perspective drawing which deserves notice. This differs from the general method above described only in the process adopted for measuring off known real lengths on lines previously laid down in perspective. In the foregoing method, a line having been once obtained in perspective, any number of points on it can be determined by having recourse to the plan. Thus in Fig. 79, 1′ V_1, the perspective of the line 1 *C* having been obtained, the points *d, c* are determined from their plans *D, C* by drawing the plans of the visual rays *s D*, *s C*, noting their intersections with *x y* and projecting up. And, similarly, any assigned real length could be measured off in perspective along 1′ V_1, by first setting off that real length along the plan 1 *C* and then proceeding as above.

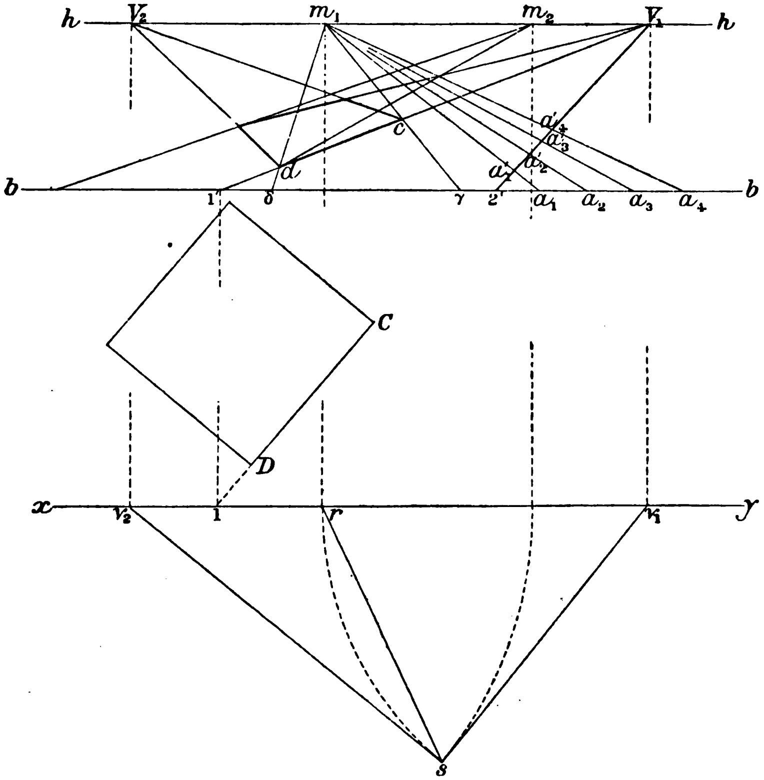
Fig. 99. There is, however, another mode of obtaining the same result, the principle of which is easily understood. In Fig. 99 (as in Fig. 79) 1^1 V_1 is the indefinite perspective of a line *D C* lying in the ground plane: it is required to obtain the perspectives of the points *D, C.* With v_1 as centre and $v_1 s$ as radius, describe the arc *s r* cutting *x y* in *r*. Project up from *r* on to *h h*, the horizon, obtaining m_1. Along *b b*, the base of the picture, set off 1′ δ equal to 1 *D.* Join m_1 δ, cutting 1′ V_1 in *d.* Then *d* is the perspective of *D.* Similarly make 1′ γ equal to 1 *C*, and join γ m_1, cutting 1′ V_1 in *c.* Then *c* is the perspective of *C.*

To prove this, join *s r*. Then it is clear that m_1 is the vanishing point for all horizontal lines parallel to *s r*. But the angle $v_1 s r$ is equal to the angle $v_1 r s$, and evidently all lines parallel to *s r* must cut *x y* and all lines parallel to $s v_1$ at equal angles. Hence all lines which vanish in m_1 must make equal angles with *b b* and with any line parallel to *D C.* Therefore, δm_1 cuts *b b* and 1′ V_1 at equal angles, and the distance 1′ δ is equal in perspective to 1′*d.* Similarly 1′γ is equal to 1′*c* and $\delta\gamma$ is equal to *d c.*

The same construction can be applied to any other line vanishing in V_1. For example, suppose that it is required to mark off successive distances of ¼″ along any line 2′ V_1. Set off 2′ a_1, $a_1 a_2$, $a_2 a_3$, ... along *b b*, each equal to ¼″. Join $a_1 m_1$, $a_2 m_1$ ... cutting 2′ V_1 in a'_1, a'_2, ... Then a'_1, a'_2 ... are perspectively ¼″ apart.

Conversely, if it were required to set off any assigned distance *beyond* or *in front of* any point d on a line $1' V_1$, join $m_1 d$ and produce it to cut $b b$ in δ. Set off the given distance along $b b$ to the *right* or *left* of δ, and join the points obtained to m_1, cutting $1' V_1$ in the points required.

FIG. 99.



Definition. "Measuring Point."

Thus a point m_1 has been found which enables any assigned distance to be set off in perspective along any line which vanishes in V_1. The point m_1 is termed a "*measuring point.*"

It follows that—

[illegible] will have a corresponding [illegible]

[illegible] horizontal lines will [illegible]

[illegible] measuring point will always be [illegible] equal to the [illegible] this distance [illegible] the vanishing [illegible] of the station [illegible]

[illegible] point for all [illegible] two measur- [illegible] indifferently, these [illegible] and left of [illegible] perpendicular [illegible]

[illegible] vanishing

point of any system of lines oblique to the ground plane [illegible] in a similar way. Thus in Fig. 100, V_1 is the vanishing point of a line inclined at a, and having $a\,e$ as its plan, the same lettering being employed as in Fig. 80. With V_1 as centre and $V_1 s_1$ as radius, describe the arc $s_1 m$ cutting the vertical through V_1 in m. Then m is the measuring point for the vanishing point V_1.

To use this measuring point, suppose that $t_1 V_1$ is the perspective of $c\,a$ indefinitely produced, and that it is required to set off along $t_1 V_1$ a distance of $\frac{1}{4}''$ measured from the picture plane. Produce $a\,c$, the plan of the line, to cut $x\,y$ in d, and project up from d on to $b\,b$, thus obtaining d'. Join $d'\,v_1$. Then $d'\,v_1$ is the perspective of the plan of the line $t_1 V_1$, and d'' (determined by projecting up from d' on to $t_1 V_1$) is the point at which the line $t_1 V_1$ cuts the picture plane. From d'' set off $d''\,e$ equal to $\frac{1}{4}''$ along the perpendicular d, d'' produced; join $e\,m$, cutting $d''\,V_1$ in q. Then $d''\,q$ is perspectively equal to $\frac{1}{4}''$.

Conversely, suppose that from any point q on a line $q\,V_1$, it is required to set off any distance—say $\frac{1}{2}''$. Produce the perspective of the plan of the line to cut the base of the picture in d'. At d' erect a perpendicular to $b\,b$. Join $m\,q$ and produce it to meet the perpendicular in e. From e set off $e\,e'$, $e\,e''$ equal to $\frac{1}{2}''$ along the perpendicular. Join $m\,e'$, $m\,e''$ cutting $q\,V_1$ in t and t_1. Then the distances $q\,t$ and $q\,t_1$ are perspectively equal to $\frac{1}{2}''$.

The above is easily proved. Join $s_1 m$. Then the angles $V_1 s_1 m$ and $V_1 m s_1$ are equal, and therefore m would be the vanishing point of a line in the same vertical plane as $a\,c$, and making with $a\,e$ the same angle as it makes with a vertical line drawn through any point of its length. Hence $d''\,q$ is equal in perspective to $d''\,e$; i.e. to $\frac{1}{4}''$. Similarly $q\,t$ is equal to $e\,e'$ and $q\,t_1$ to $e\,e''$. And, generally, if any point on any line whose perspective vanishes in V_1 be joined to m, a second line is obtained which is the perspective of a line in the same vertical plane as the first, and so placed that it cuts the first line and a vertical line drawn through any point on its own length at the same angle.

(1.) Every vanishing point will have a corresponding measuring point.

(2.) The measuring points for all horizontal lines will be on the horizon.

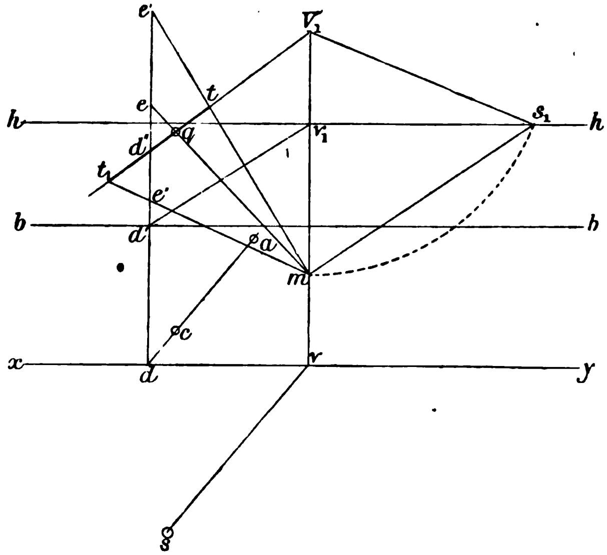
(3.) The position of the measuring point will always be at a distance from its vanishing point equal to the distance of the latter from the station point, this distance being set off along the horizon to the *left* of the vanishing point for all lines vanishing to the *right* of the station point, and *vice versâ.*

(4.) The point of sight—the vanishing point for all lines perpendicular to the picture—will have two measuring points, either of which can be used indifferently, these measuring points being on the horizon right and left of the point of sight at distances equal to the perpendicular distance of the station point from the picture.

Oblique lines.
Fig. 100.

The measuring point corresponding to the vanishing point of any system of lines oblique to the ground plane can be obtained in a similar way. Thus in Fig. 100, V_1 is

FIG. 100.



the vanishing point of a line inclined at a, and having $a\,c$ as its plan, the same lettering being employed as in Fig. 80. With V_1 as centre and $V_1\,s_1$ as radius, describe the arc $s_1\,m$ cutting the vertical through V_1 in m. Then m is the measuring point for the vanishing point V_1.

To use this measuring point, suppose that $t_1\,V_1$ is the perspective of $c\,a$ indefinitely produced, and that it is required to set off along $t_1\,V_1$ a distance of $\frac{1}{4}''$ measured from the picture plane. Produce $a\,c$, the plan of the line, to cut $x\,y$ in d, and project up from d on to $b\,b$, thus obtaining d'. Join $d'\,v_1$. Then $d'\,v_1$ is the perspective of the plan of the line $t_1\,V_1$, and d'' (determined by projecting up from d' on to $t_1\,V_1$) is the point at which the line $t_1\,V_1$ cuts the picture plane. From d'' set off $d''\,e$ equal to $\frac{1}{4}''$ along the perpendicular d, d'' produced; join $e\,m$, cutting $d''\,V_1$ in q. Then $d''\,q$ is perspectively equal to $\frac{1}{4}''$.

Conversely, suppose that from any point q on a line $q\,V_1$, it is required to set off any distance—say $\frac{1}{2}''$. Produce the perspective of the plan of the line to cut the base of the picture in d'. At d' erect a perpendicular to $b\,b$. Join $m\,q$ and produce it to meet the perpendicular in e. From e set off $e\,e'$, $e\,e''$ equal to $\frac{1}{2}''$ along the perpendicular. Join $m\,e'$, $m\,e''$ cutting $q\,V_1$ in t and t_1. Then the distances $q\,t$ and $q\,t_1$ are perspectively equal to $\frac{1}{2}''$.

The above is easily proved. Join $s_1\,m$. Then the angles $V_1\,s_1\,m$ and $V_1\,m\,s_1$ are equal, and therefore m would be the vanishing point of a line in the same vertical plane as $a\,c$, and making with $a\,c$ the same angle as it makes with a vertical line drawn through any point of its length. Hence $d''\,q$ is equal in perspective to $d''\,e$; i. e. to $\frac{1}{4}''$. Similarly $q\,t$ is equal to $e\,e'$ and $q\,t_1$ to $e\,e''$. And, generally, if any point on any line whose perspective vanishes in V_1 be joined to m, a second line is obtained which is the perspective of a line in the same vertical plane as the first, and so placed that it cuts the first line and a vertical line drawn through any point on its own length at the same angle.

The two following examples will serve to explain the practical use of measuring points.

EXAMPLE I.—A HOUSE.

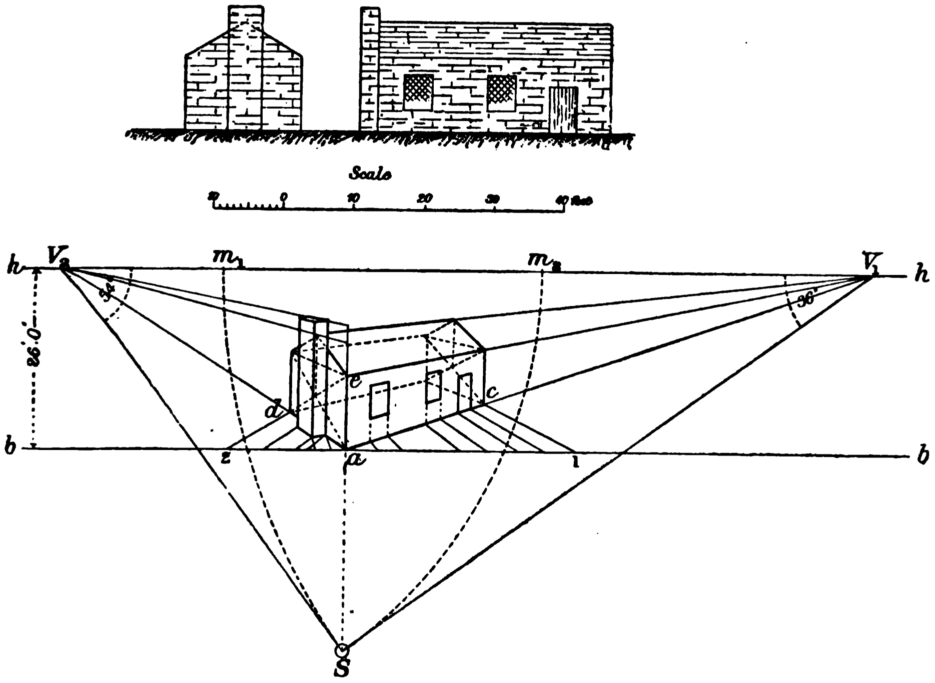
Fig. 101. In Fig. 101 the side and end elevations of a building are given: it is required to place this building in perspective, the side to make 36° with the picture plane, one corner to be *in* the latter and opposite the station point. The station point to be 55′ 0″ from the picture and 26′ 0″ above the ground plane.

It is usual to arrange the drawing somewhat differently in using the method of measuring points. The plan must of course be at hand for reference, but need not be redrawn, or traced. Similarly, the ground line need not be drawn, though its position with respect to the plan must be known or assumed. The perspective is in fact worked with the horizon and base of picture only.

In Fig. 101, *h h* and *b b* are first drawn parallel to each other and 26′ 0″ apart, measured from the scale to which the building is drawn. The station point S is then taken at a distance of 55′ 0″ from *h h*. From S draw a line $S V_1$ making 36° with *h h*; then $S V_1$ is the vanishing point for the sides of the building. Similarly $S V_2$ drawn at right angles to $S V_1$, or at 54° to *h h*, determines V_2 the vanishing point for the ends. With V_1 as centre and $V_1 S$ as radius, describe the arc $S m_1$, and with V_2 as centre and $V_2 S$ as radius, describe the arc $S m_2$; then m_1 and m_2 are the measuring points corresponding respectively to the vanishing points V_1, V_2. The perspective can now be begun.

Since the angle of the building is opposite the eye, project from S on to *b b*, and from the point *a* thus obtained draw $a V_1$, $a V_2$. From *a* set off *a* 1 along *b b* equal to the length of the side of the building, and join $1 m_1$ cutting $a V_1$ in *c*. Then *a c* is the perspective length of the side of the building. Similarly, making *a* 2 equal to the length of the end of the building and joining $2 m_2$, the length *a d* equal to the perspective of the end of the building is obtained. The perspective can now be completed, using the vertical line through *a* as the height line both from the ends and the side, and making *a e* equal to the height of the eaves. Vertical lines through the intersection of the diagonals of the ends of the building will contain the extremities of the ridge, the nearest of which is determined by setting off its height along the height line and drawing a line to V_2. The *position* of the chimney is obtained on $a\ V_2$ by using the measuring point m_2; its *projection* from the end wall by drawing lines to V_1 and using the measuring point m_1, the real distances being set off along *b b* in each case. Similarly, the positions of the windows and doors are determined by using m_1, and their heights from the height line *a e*. The construction is sufficiently shewn.

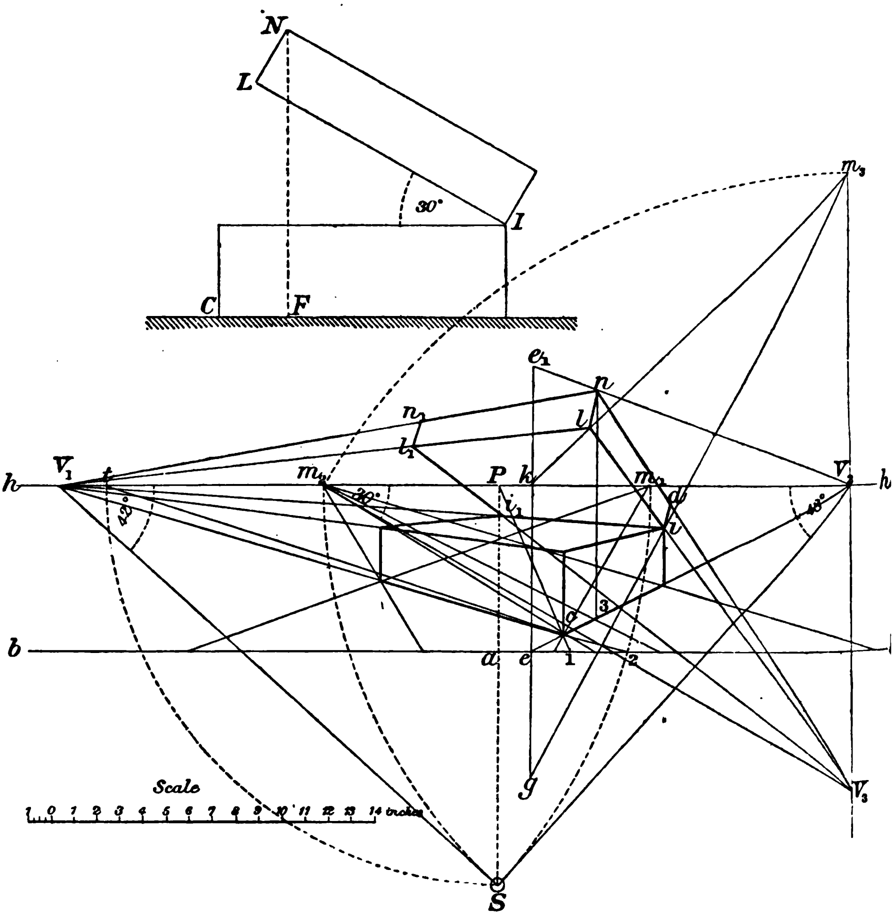
FIG. 101.



EXAMPLE II.—A BOX.

Fig. 102. In Fig. 102 the end elevation of a box with lid open at an angle of 30° is given. The length of the box is 16″. It is required to place the box in perspective, the nearest

FIG. 102.



corner to be 3″ to the right of the eye and 2¼″ from the picture plane, the front side of the box to make 42° with the latter. The eye is to be 7¼″ above the ground plane and 17″ from the picture plane.

Draw *h h* and *b b* parallel and 7¼″ apart. Take *S* at a distance of 17″ from *h h*. From *S* draw $S\ V_1$ making 42° and $S\ V_2$ making 48° with *h h*. Then V_1 and V_2 are the vanishing points for the front and end respectively. Obtain m_1 and m_2 the measuring points corresponding to V_1 and V_2. From m_2 draw a line $m_2\ V_3$ making 30° with *h h* and cutting the vertical through V_2 in V_3. With V_3 as centre and $V_3\ m_2$ as radius, describe an arc cutting this vertical in m_3. Then V_3 is the vanishing point, and m_3 the measuring point for the long sides of the ends of the inclined lid.

To determine the perspective of the nearest corner of the box, draw the perpendicular *S P* cutting *h h* in *P* and *b b* in *a*. Then *P* is the point of sight and the vanishing point for all lines at right angles to the picture. From *a*, set off *a* 1 to the right along *b b* equal to 3″. Join 1 *P*. With *P* as centre and *P s* as radius, describe an arc cutting *h h* in *t*; then *t* will be the measuring point corresponding to *P*. From 1 set off 12 along *b b* equal to 2¼″ and join 2 *t* cutting 1 *P* in *c*. Then *c* is the nearest corner of the box. From *c* draw lines to V_1 and V_2, and by using the measuring points m_1 and m_2 cut off on them the length and breadth of the box. $V_2\ c$ produced to cut *b b* in *e* will give the height line for the end of the box. The perspective of the latter can now be completed.

To obtain the lid. Join $V_3\ i$ and produce it. Join $m_3\ i$ and produce it to cut the vertical through *e* in *g*. Along the vertical set off *g k* equal to *L I* the breadth of the lid. Join $k\ m_3$ cutting $V_3\ i$ produced in *l*. Join $l\ V_1$. Then $V_3\ i_1$ produced cuts $l\ V_1$ in l_1, and the underside of the lid is completed. The vanishing and measuring points for the short sides of the ends of the lid might now be found, and these sides obtained from them, but in this case both vanishing and measuring point would lie at an inconvenient distance. The point *n* is therefore determined in another way. By using the measuring point m_2, the horizontal distance between N and C (CF) is set off from c along $c\,V_2$. A vertical line through 3 must then contain n. Make $e\,e_1$ on the height line equal to the height of N above the ground plane, and join $e_1 V_2$, cutting this vertical in n. Join $n\,V_1$ and $n\,V_3$. The points n_1 and d can be determined by their heights, the height line for the farther end of the box being drawn for the purpose of determining n_1. Or, n_1 can be obtained by drawing a vertical line through that point at the other end of the box which corresponds to 3.

The method of measuring points has one recommendation. The perspective need not take up so much room on the paper, since the plan can be on another sheet, provided that it is at hand for reference. On the other hand, it will be found that this method is more liable to lead to confusion, that it requires more lines, that if a mistake occurs it is not so easy to trace, and finally that it is not so easy to understand or to check. On the whole, therefore, although the method of measuring points lends itself to the simple geometrical objects sometimes given in examinations, it is not nearly so well suited to practical work.

To become a good perspective draughtsman requires some practice, and no mere book-reading will suffice. The student is strongly recommended, first, to work out the examples given in the foregoing pages, following the instructions and varying the conditions slightly. After this, he should attempt any other objects which appear suitable.

CHAPTER XIII.

ENGINEERING DRAWINGS.

Reference and working drawings.

ENGINEERING DRAWINGS may, roughly speaking, be divided into two classes, requiring somewhat different treatment, viz. "finished" or "reference" drawings, and "working" drawings. The former are usually drawn to comparatively small scales, and are often highly coloured and shaded, serving thus as explanatory drawings, giving a fair idea of the appearance an object may be expected to present after construction, and also forming a valuable record of work executed. The latter, on the other hand, are drawn to the largest convenient scale, often full size, or even still larger, and serve as guides to the workmen during construction.

The drawings required for a complete design of an engineering work would therefore consist of,—

1. *General views,* i. e. plans, elevations, and sections, all of which should be drawn to the same scale, and should be large enough to explain clearly the *general* arrangement of the work.

2. *Details,* i. e. plans, sections, and occasionally developments of such individual portions of the work as need further explanation as regards *form and dimensions.* The scales of these detail drawings should be as large as can conveniently be managed, and from them tracings are prepared for the use of the workmen.

General rules.

The following general rules should be observed in all Engineering Drawings:

Rule 1.

A good drawing should have a well-adjusted balance in all its parts. Thus the thickness of the lines should be proportioned to the scale, and the general depth of colour should suit the lines, while the printing should be of such size and boldness as to harmonize with the lines and colour, neither overpowering the latter nor being thrown into shade by them. It will be evident, therefore, that in making a really good drawing a considerable amount of artistic feeling and good taste is called into play.

Rule 2. In drawing out details, three views of each object will be generally necessary, and more views will be required when its opposite sides present any differences of appearance. Sections should also be made on all centre lines or lines of symmetry, whether horizontal or vertical. If the object is *absolutely* symmetrical about any line, a half section and half elevation may be drawn and combined.

Rule 3. All views and sections should be correctly drawn so as to agree exactly, and in arranging such views on the paper, care should be taken that where possible, elevations, plans, and sections should be projected from each other, so that their connection and correspondence may be readily traced.

Rule 4. In commencing a drawing, the "centre lines" of each part, i. e. the lines about which that part is symmetrical, should be laid down on paper first, each view or section being afterwards completed by building on to its centre line. In thus laying down centre lines which are to fix upon the paper the position of each view, care should be taken that this position is properly chosen, both with regard to Rule 3, and also in order that the whole sheet of views when complete, may, as far as possible, have a symmetrical and well-balanced appearance.

Rule 5. A scale should be shewn with each drawing. Where different portions are of different scales, these scales should be all drawn, and the views to which they refer noted.

Rule 6. In working drawings, *every part should have its dimension clearly figured,* small arrow-heads being used to indicate the extent of the dimension. When many dimensions cross each other, dotted black lines or fine continuous red lines should be drawn connecting the arrow-heads, the written dimension being inserted in a small gap left at or near the centre of the lines. Black figuring with red lines looks very well.

The following general hints will also be found useful.

Pencilling. The pencilling should be very carefully made; all mistakes should be corrected, and all superfluous lines or portions of lines removed by indiarubber or soft eraser before inking in. The pencils used should have fine points, but should not be too hard. Chisel-pointed pencils are not desirable for general drawing. In setting off dimensions, it is better, if possible, to use a scale reading to the edge; such a scale can always be drawn on paper, and a clean edge ensured by cutting it off with a sharp knife. When dividers are used, their points should not be forced through the paper, but gently pressed, so as to leave merely a slight mark, which can be made more conspicuous, if required, by a pencil circle round it.

Ink lines. It is generally best to pencil the whole of a drawing before inking it in, as uniformity of the ink lines is more likely to be obtained; and, moreover, since all the parts of a drawing are mutually dependent, corrections in its earlier portions are frequently necessary as the more advanced are constructed. In fact, no view can be considered as absolutely correct till all the others are complete.

All ink lines should be firm and perfectly black: no line should be so fine as to become ragged after the drawing has been cleaned with bread or indiarubber. As ink lines lose much of their brightness when rubbed in cleaning up, it is sometimes advisable, where time is not an object, to ink in a drawing, clean it with bread, sponge it over with clean water, then lay on the colours, and, lastly, go over all the lines again. The sponging removes all superfluous ink and prevents the lines from running, while it at the same time prepares the paper to receive colour; and the lines being redrawn *after* the cleaning up, will be quite bright and distinct.

Chain dotting. Chain dotted lines are better than ordinary dotted lines, and require less care, since the latter must be very even and uniform to look well.

Shade-lines.

It is best to add the shade-lines after colouring, for the reasons above stated. They should always have their thickness on the *outside* of the true outline of the shade-lined portion. Thus, if a drawing is to be afterwards "scaled," the dimensions would be taken to the *inner edges* of the shade-lines. In shade-lining a drawing the rules laid down in Chapter X. should be rigidly adhered to. Shade-lines which do not strictly follow recognised rules are worse than useless. It should also be remembered that shade-lines are used to render a drawing more intelligible, and not to hide defects in its execution.

Colour.

Colour is a serviceable ally, but a dangerous enemy to the draughtsman: unless therefore he has a natural eye for its harmony and some artistic taste, it is far better that it should be only sparingly employed.

Conventional colours.

Conventional colours, serving to distinguish the different materials, vary considerably; drawing offices frequently following tradition in the matter. Drawings for government purposes, however, are usually coloured according to fixed colour tables.

The following colours are recommended for the more common materials, viz.:—

1. Cast iron	.. .. ..	Payne's grey, or a mixture of indigo, sepia, and lake.
2. Wrought iron	.. ..	Prussian blue.
3. Steel	.. ..	Prussian blue and crimson lake.
4. Brass	Elevation ..	Gamboge.
	Section	Dark Indian yellow.
5. Copper	.. ..	Gamboge and crimson lake.
6. Ordinary building stone		Sepia and yellow ochre, or Roman ochre and a little lake.
7. Brick	Elevation ..	Light red, with a little carmine.
	Section	Scarlet lake, or carmine.
8. Wood	Elevation ..	Yellow ochre, or raw-sienna.
	Section	Burnt-sienna.
9. Earth	.. .. ..	Burnt-umber.
10. Lines	Red	Carmine, or scarlet lake.
	Blue	French ultramarine.

Flat washes.

To lay on a flat wash requires some little practice; failure is generally the result of neglecting the following rules: 1st. Take care that the brush and saucer are perfectly clean. The former should be a large red sable with a fine point. 2nd. Slope the drawing board slightly. 3rd. Lay on enough colour just to cover the paper, and never leave small pools of colour on any portion. Take care not to have too much, nor too little colour in the brush. The latter should, in fact, never be so full that perfect command of the colour is lost. 4th. Some earthy colours form a sediment when mixed with water: this sediment should be allowed to settle, and the amount of colour required in the brush should either be skimmed off the surface without stirring the contents of the saucer, or the latter should be thoroughly stirred each time a fresh brush-full is taken. 5th. On finishing off a wash, remove any superfluous colour with the brush, having first emptied the latter by means of a piece of blotting-paper. 6th. Always try colours on a piece of waste paper before commencing, but remember that the general effect of a large wash cannot be judged from that of a mere strip.

Cast shadows and shading.

Cast shadows and shading should be laid on in a neutral tint *before colouring*, and the colours afterwards glazed over them. This gives softness to the general effect. Indian ink answers very well for shading, though other neutral tints may be used.

Shading machinery.

In shading details of machinery a good striking effect is produced by slightly shading the darkest portions only, in the colour proper to the material. In this way, with very little trouble, the appearance of roundness can be obtained and the nature of the material at the same time indicated. Cast shadows, i. e. the shadows of one part on another, may be altogether omitted, since their correct determination will often be a somewhat tedious process.

Colouring sections.

Coloured sections should always be made either *decidedly darker, or decidedly lighter* than either plans or elevations. The former effect, which is the most usual, may be produced either by merely darkening the wash or by diagonal cross lining laid on either with a brush or ruling pen. These cross lines should be in the colour proper to the section; they should be uniform and at equal distances apart, and should be inclined at 45° to the edges of the paper. When no colour is employed, varied cross lining in ink serves to distinguish materials.

Light edges.

In colouring sections, the wash should not be carried quite to the bounding line of those edges which are toward the light (see Chapter X.), but a narrow strip, say about $\frac{1}{32}''$, should be left white. This, with the aid of shade-lines to the opposite edges, gives considerable relief to the sections, making them appear to stand up from the paper. When a complete section consists of several contiguous sectional parts, it will not usually be necessary to leave lights on the edges of *each* part. But in a case in which it is particularly desirable that *each* sectional part should be clearly distinguished from those in contact with it, light strips should be left on the proper light edges of each part. Of the latter case, the section of a joint of a wrought-iron bridge or roof affords an excellent example. In this instance light strips to every light edge are most valuable, showing at a glance the extent and form of each portion. It will usually be best, if the strip to be left white is a long one, to rule a line parallel to the edge of the section marking the edge of the wash. Such a line may be ruled in pencil, or, better still, in the colour of the wash which is to follow. When the sectional part is very narrow, as will frequently occur in the case of wrought-iron plates, &c., the colour may be ruled in with a single broad line, the light strip being left at the same time.

Sections through the axis of a bolt or rod.

When a section is made by a plane passing through the axis of a rivet, bolt shaft, spindle, or any kind of rod of a circular section, the latter is not considered to be cut by the section plane, but is treated as a plan or elevation.

Invisible portions.

It will sometimes be desirable to indicate the form and position of the invisible portions of an object; that is, of those portions which cannot be seen from the position supposed to be occupied by the spectator. These invisible portions should be shewn by dotted lines.

Working drawings. In working drawings, as a rule, *sections only need be coloured.* Cases will, however, arise in which colour on plans and elevations is desirable. In drawings of this class clearness as to arrangement, form, dimensions, and material is of the first importance. Where any explanation is needed, it should be appended as a note, written on or by the side of the part referred to.

Printing. The printing should be so arranged that it can all be read without altering the position of the drawing. Printing requires considerable practice, and it will be better usually to adopt two styles only, viz. upright block capitals and sloping italic capitals and small letters.

Large block capitals. No. 1, Pl. XX. Alphabet No. 1, Pl. XX., consists of large upright block capitals: these should be used for the principal headings of drawings. The form and dimensions shewn should be preserved, but in accordance with Rule 1, *p.* 165, the letters may be varied in several ways, to suit the general effect of the drawing. Thus the strokes may be all black, as shewn in Pl. XX: they may be put in with light Indian ink, or they may be left in outline without filling in, and in this latter case they should be shade-lined as if in elevation. Again, the fine lines may be omitted altogether, the shade-lines alone being retained (D, E, F, Pl. XX, No. 1). Roman numerals should be used with this alphabet.

Sloping block capitals. Sloping block capitals may, however, be used if preferred. (W, X, Y, Z, Pl. XX., No. 1.)

Small block capitals, No. 2, Pl. XX. Alphabet No. 2, Pl. XX., consists of small block capitals: their *form* should be exactly the same as that of Alphabet No. 1. These small block letters should be used for minor headings, i. e. for headings applying only to some particular part of a drawing, as for instance, "Section on A B," "Side elevation," &c. Ordinary or Arabic numerals should be used with these letters. Sloping small block capitals (H P Q R, Pl. XX., No. 2) may be used if preferred.

Italic letters, No. 3, Pl. XX.

Alphabet No. 3, Pl. XX., consists of sloping or italic capitals, small letters, and numerals. These should be used on drawings for explanatory notes, for the names of the different parts of a structure, and for dimensions. The size of these letters may vary somewhat, according to the purpose for which they are used. The tall small letters, such as *d* and *l*, and all numerals used with them, should have heights equal to that of the capitals.

It is hoped that the hints contained in the preceding pages will be found useful. No written instructions can in a subject like drawing take the place of practice and experience, but attention to the former will at least help to lessen the number of those vexatious and often needless failures which so frequently discourage beginners.

CHAPTER XIV.

SELECTION OF INSTRUMENTS, ETC.

Needle points. NEEDLE-POINTED INSTRUMENTS are now in general use. They are much to be preferred to ordinary instruments, but they require more careful treatment and a lighter hand.

In choosing needle-pointed instruments, be sure that the "shoulder" from which the needle projects is sufficiently large. This shoulder has to rest on the paper when the instrument is in use, and if it is too small it is apt to make a hole.

In choosing all instruments look well to the workmanship. See that the movable limbs of the compasses fit properly, and that all joints work smoothly.

Bows. The "bows," both pen and pencil, should be double-jointed, so that when in use both limbs may be bent so as to stand at right angles to the plane of the paper. If this cannot be done, enlargement of the hole takes place, and the pen-bow cannot be made to give an even arc.

Spring bows. See that the springs of the "spring bows" are sufficiently strong. If this is not the case, the pen or pencil leg will draw in to the centre, and a crooked spiral instead of a circle will result. This is especially likely to occur when very small circles are being drawn. The spring pen-bow should have a joint in the pen-leg, the adjusting screw being attached *below* the joint. This arrangement enables the leg in question to be kept approximately at right angles to the plane of the paper.

Hair dividers. The spring of the adjustable limb of the "hair dividers" should be strong, as it is this spring alone which prevents the points from spreading. That form of hair dividers in which the adjustable limb is rendered immovable in any position is to be preferred. In this instrument the milled head of the adjusting screw works in a guard attached outside the limb, and gives the motion in *both* directions. If, however, the joint of the dividers works smoothly, the hair adjustment will rarely be required.

Napier compasses.

Napier compasses, from their extreme portability, are very useful; they form a set of instruments in themselves. Those in which the limbs can be lengthened by a telescopic arrangement are to be preferred. The latter should be strong enough to be perfectly rigid without being inconveniently heavy.

Rigidity in all instruments of this class is an important qualification; looseness of joints is fatal.

Proportional compasses.

Proportional compasses are occasionally useful, but their graduation cannot always be depended upon. Those in which the points are turned up at right angles are best, as they are not affected by the shortening which often takes place when the instrument is sharpened.

Drawing pens.

The choice of a drawing pen requires some experience. Drawing pens are made either with two spring nibs, or with one of the nibs hinged and pressed outwards by a small spring. The former pens are best, but the latter are easier to clean and sharpen. In either case the spring should be a strong one. Pens in which the nibs are much bowed, i. e. curve in much, should be rejected. A good pen should have broad, thick, and somewhat short nibs, containing plenty of metal, which can always be worked down to any extent in sharpening. The nibs should be of exactly the same length and form, and with a gradual and nearly straight taper. The points should be rounded off, so as to present no sharp angle. They should not be too sharp, or they will cut the surface of the paper, and are apt to run away from the straight-edge; nor too blunt, or they will not give a decently fine line.

Sharpening drawing pens.

A drawing pen should be sharpened as follows. Screw up till the nibs are in contact. Work the pen gently on a stone with a little oil, holding it so as to be always in a plane at right angles to that of the stone. Proceeding in this way the proper *form* is given to the points, and they are moreover kept alike and of equal length. The points should be looked at (through a magnifying glass, if required) at short intervals. When the proper form has been obtained, open the points a little and proceed to sharpen each nib separately, taking the greatest care not to spoil the previously imparted shape, and endeavouring to give *exactly the same* degree of sharpness to both nibs. It remains merely to remove the "burr" left by the previous operations. This should be done very carefully and with very fine emery paper. When using the latter *between* the nibs, be particularly careful to remove the burr only, and not any of the metal of the points, as if an outward splay is given to the latter, the pen will not work at all.

If the above directions are followed, it should be possible with a little practice to succeed in obtaining a pen which will give a firm, smooth, and uniform line. Every draughtsman should be able to sharpen his own pens, as the latter frequently leave the maker's hands unfit for use; and in any case, if a pen is much used it will require constant attention.

In using a drawing pen, avoid pressing it too hard against the straight-edge, and keep the second finger on the milled head of the screw, and not on the outer nib itself.

Dotting pens.

Dotting pens are not recommended: they give a great deal of trouble, and are very uncertain in their action.

Parallel rulers.

Parallel rulers should be of the "*rolling*" class, not less than 15″ in length, made of metal, and the heavier the better. Light wooden rolling rulers, are very untrustworthy. *Jointed* parallel rulers should not be used.

LONDON: PRINTED BY WILLIAM CLOWES AND SONS, LIMITED, STAMFORD STREET
AND CHARING CROSS.

BOOKS RELATING

TO

APPLIED SCIENCE

PUBLISHED BY

E. & F. N. SPON,

LONDON: 16, CHARING CROSS.

NEW YORK: 35, MURRAY STREET.

A Pocket-Book for Chemists, Chemical Manufacturers, Metallurgists, Dyers, Distillers, Brewers, Sugar Refiners, Photographers, Students, etc., etc. By THOMAS BAYLEY, Assoc. R.C. Sc. Ireland, Analytical and Consulting Chemist and Assayer. Second edition, with additions, 437 pp., royal 32mo, roan, gilt edges, 5*s*.

SYNOPSIS OF CONTENTS:

Atomic Weights and Factors—Useful Data—Chemical Calculations—Rules for Indirect Analysis—Weights and Measures—Thermometers and Barometers—Chemical Physics—Boiling Points, etc.—Solubility of Substances—Methods of Obtaining Specific Gravity—Conversion of Hydrometers—Strength of Solutions by Specific Gravity—Analysis—Gas Analysis—Water Analysis—Qualitative Analysis and Reactions—Volumetric Analysis—Manipulation—Mineralogy—Assaying—Alcohol—Beer—Sugar—Miscellaneous Technological matter relating to Potash, Soda, Sulphuric Acid, Chlorine, Tar Products, Petroleum, Milk, Tallow, Photography, Prices, Wages, Appendix, etc., etc.

The Mechanician: A Treatise on the Construction and Manipulation of Tools, for the use and instruction of Young Engineers and Scientific Amateurs, comprising the Arts of Blacksmithing and Forging; the Construction and Manufacture of Hand Tools, and the various Methods of Using and Grinding them; the Construction of Machine Tools, and how to work them; Machine Fitting and Erection; description of Hand and Machine Processes; Turning and Screw Cutting; principles of Constructing and details of Making and Erecting Steam Engines, and the various details of setting out work, etc., etc. By CAMERON KNIGHT, Engineer. *Containing* 1147 *illustrations*, and 397 pages of letter-press. Third edition, 4to, cloth, 18*s*.

On Designing Belt Gearing. By E. J. COWLING WELCH, Mem. Inst. Mech. Engineers, Author of 'Designing Valve Gearing.' Fcap. 8vo, sewed, 6*d.*

A Handbook of Formulæ, Tables, and Memoranda, for Architectural Surveyors and others engaged in Building. By J. T. HURST, C.E. Thirteenth edition, royal 32mo, roan, 5*s.*

"It is no disparagement to the many excellent publications we refer to, to say that in our opinion this little pocket-book of Hurst's is the very best of them all, without any exception. It would be useless to attempt a recapitulation of the contents, for it appears to contain almost *everything* that anyone connected with building could require, and, best of all, made up in a compact form for carrying in the pocket, measuring only 5 in. by 3 in., and about ¾ in. thick, in a limp cover. We congratulate the author on the success of his laborious and practically compiled little book, which has received unqualified and deserved praise from every professional person to whom we have shown it."—*The Dublin Builder.*

The Cabinet Maker; being a Collection of the most approved designs in the Mediæval, Louis-Seize, and Old English styles, for the use of Cabinet Makers, Carvers, &c. By R. CHARLES. 96 *plates*, folio, half-bound, 10*s.* 6*d.*

Quantity Surveying. By J. LEANING. With 42 illustrations, crown 8vo, cloth, 9*s.*

CONTENTS:

A complete Explanation of the London Practice.
General Instructions.
Order of Taking Off.
Modes of Measurement of the various Trades.
Use and Waste.
Ventilation and Warming.
Credits, with various Examples of Treatment.
Abbreviations.
Squaring the Dimensions.
Abstracting, with Examples in illustration of each Trade.
Billing.
Examples of Preambles to each Trade.
Form for a Bill of Quantities.
Do. Bill of Credits.
Do. Bill for Alternative Estimate.
Restorations and Repairs, and Form of Bill.
Variations before Acceptance of Tender.
Errors in a Builder's Estimate.
Schedule of Prices.
Form of Schedule of Prices.
Analysis of Schedule of Prices.
Adjustment of Accounts.
Form of a Bill of Variations.
Remarks on Specifications.
Prices and Valuation of Work, with Examples and Remarks upon each Trade.
The Law as it affects Quantity Surveyors, with Law Reports.
Taking Off after the Old Method.
Northern Practice.
The General Statement of the Methods recommended by the Manchester Society of Architects for taking Quantities.
Examples of Collections.
Examples of "Taking Off" in each Trade.
Remarks on the Past and Present Methods of Estimating.

A Practical Treatise on Heat, as applied to the Useful Arts; for the Use of Engineers, Architects, &c. By THOMAS BOX. *With* 14 *plates.* Third edition, crown 8vo, cloth, 12*s.* 6*d.*

A Descriptive Treatise on Mathematical Drawing Instruments: their construction, uses, qualities, selection, preservation, and suggestions for improvements, with hints upon Drawing and Colouring. By W. F. STANLEY, M.R.I. Fifth edition, *with numerous illustrations*, crown 8vo, cloth, 5*s.*

Spons' Architects' and Builders' Pocket-Book of Prices and Memoranda. Edited by W. Young, Architect. Royal 32mo, roan, 4*s*. 6*d*.; or cloth, red edges, 3*s*. 6*d*. *Published annually.* Tenth edition. *Now ready.*

Long-Span Railway Bridges, comprising Investigations of the Comparative Theoretical and Practical Advantages of the various adopted or proposed Type Systems of Construction, with numerous Formulæ and Tables giving the weight of Iron or Steel required in Bridges from 300 feet to the limiting Spans; to which are added similar Investigations and Tables relating to Short-span Railway Bridges. Second and revised edition. By B. Baker, Assoc. Inst. C.E. *Plates,* crown 8vo, cloth, 5*s*.

Elementary Theory and Calculation of Iron Bridges and Roofs. By August Ritter, Ph.D., Professor at the Polytechnic School at Aix-la-Chapelle. Translated from the third German edition, by H. R. Sankey, Capt. R.E. With 500 *illustrations,* 8vo, cloth, 15*s*.

The Builder's Clerk: a Guide to the Management of a Builder's Business. By Thomas Bales. Fcap. 8vo, cloth, 1*s*. 6*d*.

The Elementary Principles of Carpentry. By Thomas Tredgold. Revised from the original edition, and partly re-written, by John Thomas Hurst. Contained in 517 pages of letterpress, and *illustrated with* 48 *plates and* 150 *wood engravings.* Third edition, crown 8vo, cloth, 18*s*.

Section I. On the Equality and Distribution of Forces—Section II. Resistance of Timber—Section III. Construction of Floors—Section IV. Construction of Roofs—Section V. Construction of Domes and Cupolas—Section VI. Construction of Partitions—Section VII. Scaffolds, Staging, and Gantries—Section VIII. Construction of Centres for Bridges—Section IX. Coffer-dams, Shoring, and Strutting—Section X. Wooden Bridges and Viaducts—Section XI. Joints, Straps, and other Fastenings—Section XII. Timber.

Our Factories, Workshops, and Warehouses: their Sanitary and Fire-Resisting Arrangements. By B. H. Thwaite, Assoc. Mem. Inst. C.E. *With* 183 *wood engravings,* crown 8vo, cloth, 9*s*.

Gold: Its Occurrence and Extraction, embracing the Geographical and Geological Distribution and the Mineralogical Characters of Gold-bearing rocks; the peculiar features and modes of working Shallow Placers, Rivers, and Deep Leads; Hydraulicing; the Reduction and Separation of Auriferous Quartz; the treatment of complex Auriferous ores containing other metals; a Bibliography of the subject and a Glossary of Technical and Foreign Terms. By Alfred G. Lock, F.R.G.S. *With numerous illustrations and maps,* 1250 pp., super-royal 8vo, cloth, 2*l*. 12*s*. 6*d*.

Progressive Lessons in Applied Science. By EDWARD SANG, F.R.S.E. Crown 8vo, cloth, each Part, 3*s.*

Part 1. Geometry on Paper—Part 2. Solidity, Weight, and Pressure—Part 3. Trigonometry, Vision, and Surveying Instruments.

A Practical Treatise on Coal Mining. By GEORGE G. ANDRÉ, F.G.S., Assoc. Inst. C.E., Member of the Society of Engineers. *With* 82 *lithographic plates.* 2 vols., royal 4to, cloth, 3*l.* 12*s.*

Sugar Growing and Refining: a Comprehensive Treatise on the Culture of Sugar-yielding Plants, and the Manufacture, Refining, and Analysis of Cane, Beet, Maple, Milk, Palm, Sorghum, and Starch Sugars, with copious statistics of their production and commerce, and a chapter on the distillation of Rum.. By CHARLES G. WARNFORD LOCK, F.L.S., &c., and G. W. WIGNER and R. H. HARLAND, FF.C.S., FF.I.C. *With* 205 *illustrations,* 8vo, cloth, 30*s.*

Spons' Information for Colonial Engineers. Edited by J. T. HURST. Demy 8vo, sewed.

No. 1, Ceylon. By ABRAHAM DEANE, C.E. 2*s.* 6*d.*

CONTENTS:

Introductory Remarks—Natural Productions—Architecture and Engineering—Topography, Trade, and Natural History—Principal Stations—Weights and Measures, etc., etc.

No. 2. Southern Africa, including the Cape Colony, Natal, and the Dutch Republics. By HENRY HALL, F.R.G.S., F.R.C.I. With Map. 3*s.* 6*d.*

CONTENTS:

General Description of South Africa—Physical Geography with reference to Engineering Operations—Notes on Labour and Material in Cape Colony—Geological Notes on Rock Formation in South Africa—Engineering Instruments for Use in South Africa—Principal Public Works in Cape Colony: Railways, Mountain Roads and Passes, Harbour Works, Bridges, Gas Works, Irrigation and Water Supply, Lighthouses, Drainage and Sanitary Engineering, Public Buildings, Mines—Table of Woods in South Africa—Animals used for Draught Purposes—Statistical Notes—Table of Distances—Rates of Carriage, etc.

No. 3. India. By F. C. DANVERS, Assoc. Inst. C.E. With Map. 4*s.* 6*d.*

CONTENTS:

Physical Geography of India—Building Materials—Roads—Railways—Bridges—Irrigation—River Works—Harbours—Lighthouse Buildings—Native Labour—The Principal Trees of India—Money—Weights and Measures—Glossary of Indian Terms, etc.

A Practical Treatise on Casting and Founding, including descriptions of the modern machinery employed in the art. By N. E. SPRETSON, Engineer. Third edition, with 82 *plates* drawn to scale, 412 pp., demy 8vo, cloth, 18*s.*

The Clerk of Works: a Vade-Mecum for all engaged in the Superintendence of Building Operations. By G. G. HOSKINS, F.R.I.B.A. Third edition, fcap. 8vo, cloth, 1*s.* 6*d.*

Tropical Agriculture; or, the Culture, Preparation, Commerce, and Consumption of the Principal Products of the Vegetable Kingdom, as furnishing Food, Clothing, Medicine, etc., and in their relation to the Arts and Manufactures; forming a practical treatise and Handbook of Reference for the Colonist, Manufacturer, Merchant, and Consumer, on the Cultivation, Preparation for Shipment, and Commercial Value, etc., of the various Substances obtained from Trees and Plants entering into the Husbandry of Tropical and Sub-Tropical Regions. By P. L. SIMMONDS. Second edition, revised and improved, 515 pages, 8vo, cloth, 1*l.* 1*s.*

Steel: its History, Manufacture, and Uses. By J. S. JEANS, Secretary of the Iron and Steel Institute. 860 pages and 24 *plates,* 8vo, cloth, 36*s.*

American Foundry Practice: Treating of Loam, Dry Sand, and Green Sand Moulding, and containing a Practical Treatise upon the Management of Cupolas, and the Melting of Iron. By T. D. WEST, Practical Iron Moulder and Foundry Foreman. Second edition, *with numerous illustrations,* crown 8vo, cloth, 10*s.* 6*d.*

The Maintenance of Macadamised Roads. By T. CODRINGTON, M.I.C.E, F.G.S., General Superintendent of County Roads for South Wales. 8vo, cloth, 6*s.*

Hydraulic Steam and Hand Power Lifting and Pressing Machinery. By FREDERICK COLYER, M. Inst. C.E., M. Inst. M.E. *With* 73 *plates,* 8vo, cloth, 18*s.*

Pumps and Pumping Machinery. By F. COLYER, M.I.C.E., M.I.M.E. *With* 23 *folding plates,* 8vo, cloth, 12*s.* 6*d.*

Tables of the Principal Speeds occurring in Mechanical Engineering, expressed in metres in a second. By P. KEERAYEFF, Chief Mechanic of the Obouchoff Steel Works, St. Petersburg; translated by SERGIUS KERN, M.E. Fcap. 8vo, sewed, 6*d.*

Girder Making and the Practice of Bridge Building in Wrought Iron, illustrated by Examples of Bridges, Piers, and Girder Work, etc., constructed at the Skerne Iron Works, Darlington, by EDWARD HUTCHINSON, M. Inst. M.E. *With* 35 *plates,* demy 8vo, cloth, 10*s.* 6*d.*

Spons' Dictionary of Engineering, Civil, Mechanical, Military, and Naval; with technical terms in French, German, Italian, and Spanish, 3100 pp., and *nearly* 8000 *engravings*, in super-royal 8vo, in 8 divisions, 5*l.* 8*s.* Complete in 3 vols., cloth, 5*l.* 5*s.* Bound in a superior manner, half-morocco, top edge gilt, 3 vols., 6*l.* 12*s.*

See page 15.

A Treatise on the Origin, Progress, Prevention, and Cure of Dry Rot in Timber; with Remarks on the Means of Preserving Wood from Destruction by Sea-Worms, Beetles, Ants, etc. By THOMAS ALLEN BRITTON, late Surveyor to the Metropolitan Board of Works, etc., etc. *With* 10 *plates*, crown 8vo, cloth, 7*s.* 6*d.*

Metrical Tables. By G. L. MOLESWORTH, M.I.C.E. 32mo, cloth, 1*s.* 6*d.*

CONTENTS.

General—Linear Measures—Square Measures—Cubic Measures—Measures of Capacity—Weights—Combinations—Thermometers.

A Handbook of Electrical Testing. By H. R. KEMPE, Member of the Society of Telegraph Engineers. New edition, revised and enlarged, *with* 81 *illustrations.* Crown 8vo, cloth, 12*s.* 6*d.*

Electro-Telegraphy. By FREDERICK S. BEECHEY, Telegraph Engineer. A Book for Beginners. *Illustrated.* Fcap. 8vo, sewed, 6*d.*

Handrailing: by the Square Cut. By JOHN JONES, Staircase Builder. Fourth edition, *with seven plates*, 8vo, cloth, 3*s.* 6*d.*

Handrailing: by the Square Cut. By JOHN JONES, Staircase Builder. Part Second, *with eight plates*, 8vo, cloth, 3*s.* 6*d.*

The Gas Consumer's Handy Book. By WILLIAM RICHARDS, C.E. *Illustrated.* 18mo, sewed, 6*d.*

Steam Heating for Buildings; or, Hints to Steam Fitters, being a description of Steam Heating Apparatus for Warming and Ventilating Private Houses and large Buildings; with Remarks on Steam, Water, and Air in their relation to Heating; to which are added miscellaneous Tables. By J. W. BALDWIN, Steam Heating Engineer. *With many illustrations.* Second edition, crown 8vo, cloth, 10*s.* 6*d.*

A Pocket-Book of Useful Formulæ and Memoranda

for Civil and Mechanical Engineers. By GUILFORD L. MOLESWORTH, Mem. Inst. C.E., Consulting Engineer to the Government of India for State Railways. *With numerous illustrations,* 744 pp. Twenty-first edition, revised and enlarged, 32mo, roan, 6*s*.

SYNOPSIS OF CONTENTS:

Surveying, Levelling, etc.—Strength and Weight of Materials—Earthwork, Brickwork, Masonry, Arches, etc.—Struts, Columns, Beams, and Trusses—Flooring, Roofing, and Roof Trusses—Girders, Bridges, etc.—Railways and Roads—Hydraulic Formulæ—Canals, Sewers, Waterworks, Docks—Irrigation and Breakwaters—Gas, Ventilation, and Warming—Heat, Light, Colour, and Sound—Gravity: Centres, Forces, and Powers—Millwork, Teeth of Wheels, Shafting, etc.—Workshop Recipes—Sundry Machinery—Animal Power—Steam and the Steam Engine—Water-power, Water-wheels, Turbines, etc.—Wind and Windmills—Steam Navigation, Ship Building, Tonnage, etc.—Gunnery, Projectiles, etc.—Weights, Measures, and Money—Trigonometry, Conic Sections, and Curves—Telegraphy—Mensuration—Tables of Areas and Circumference, and Arcs of Circles—Logarithms, Square and Cube Roots, Powers—Reciprocals, etc.—Useful Numbers—Differential and Integral Calculus—Algebraic Signs—Telegraphic Construction and Formulæ.

Spons' Tables and Memoranda for Engineers;

selected and arranged by J. T. HURST, C.E., Author of 'Architectural Surveyors' Handbook,' 'Hurst's Tredgold's Carpentry,' etc. Fifth edition, 64mo, roan, gilt edges, 1*s*.; or in cloth case, 1*s*. 6*d*.

This work is printed in a pearl type, and is so small, measuring only 2½ in. by 1¼ in. by ¼ in. thick, that it may be easily carried in the waistcoat pocket.

"It is certainly an extremely rare thing for a reviewer to be called upon to notice a volume measuring but 2½ in. by 1¾ in., yet these dimensions faithfully represent the size of the handy little book before us. The volume—which contains 118 printed pages, besides a few blank pages for memoranda—is, in fact, a true pocket-book, adapted for being carried in the waistcoat pocket, and containing a far greater amount and variety of information than most people would imagine could be compressed into so small a space. . . . The little volume has been compiled with considerable care and judgment, and we can cordially recommend it to our readers as a useful little pocket companion."—*Engineering*.

Analysis, Technical Valuation, Purification and Use

of Coal Gas. By the Rev. W. R. BOWDITCH, M.A. *With wood engravings,* 8vo, cloth, 12*s*. 6*d*.

Condensation of Gas—Purification of Gas—Light—Measuring—Place of Testing Gas—Test Candles—The Standard for Measuring Gas-light—Test Burners—Testing Gas for Sulphur—Testing Gas for Ammonia—Condensation by Bromine—Gravimetric Method of taking Specific Gravity of Gas—Carburetting or Naphthalizing Gas—Acetylene—Explosions of Gas—Gnawing of Gaspipes by Rats—Pressure as related to Public Lighting, etc.

A Practical Treatise on Natural and Artificial

Concrete, its Varieties and Constructive Adaptations. By HENRY REID, Author of the 'Science and Art of the Manufacture of Portland Cement.' New Edition, *with* 59 *woodcuts and* 5 *plates,* 8vo, cloth, 15*s*.

Hydrodynamics: Treatise relative to the Testing of

Water-Wheels and Machinery, with various other matters pertaining to Hydrodynamics. By JAMES EMERSON. *With numerous illustrations,* 360 pp. Third edition, crown 8vo, cloth, 4*s*. 6*d*.

The Gas Analyst's Manual. By F. W. HARTLEY, Assoc. Inst. C.E., etc. *With numerous illustrations.* Crown 8vo, cloth, 6*s.*

Gas Measurement and Gas Meter Testing. By F. W. HARTLEY. Fourth edition, revised and extended. *Illustrated,* crown 8vo, cloth, 4*s.*

The French-Polisher's Manual. By a French-Polisher; containing Timber Staining, Washing, Matching, Improving, Painting, Imitations, Directions for Staining, Sizing, Embodying, Smoothing, Spirit Varnishing, French-Polishing, Directions for Re-polishing. Third edition, royal 32mo, sewed, 6*d.*

Hops, their Cultivation, Commerce, and Uses in various Countries. By P. L. SIMMONDS. Crown 8vo, cloth, 4*s.* 6*d.*

A Practical Treatise on the Manufacture and Distribution of Coal Gas. By WILLIAM RICHARDS. Demy 4to, with *numerous wood engravings and* 29 *plates,* cloth, 28*s.*

SYNOPSIS OF CONTENTS:

Introduction—History of Gas Lighting—Chemistry of Gas Manufacture, by Lewis Thompson, Esq., M.R.C.S.—Coal, with Analyses, by J. Paterson, Lewis Thompson, and G. R. Hislop, Esqrs.—Retorts, Iron and Clay—Retort Setting—Hydraulic Main—Condensers—Exhausters—Washers and Scrubbers—Purifiers—Purification—History of Gas Holder—Tanks, Brick and Stone, Composite, Concrete, Cast-iron, Compound Annular Wrought-iron—Specifications—Gas Holders—Station Meter—Governor—Distribution—Mains—Gas Mathematics, or Formulæ for the Distribution of Gas, by Lewis Thompson, Esq.—Services—Consumers' Meters—Regulators—Burners—Fittings—Photometer—Carburization of Gas—Air Gas and Water Gas—Composition of Coal Gas, by Lewis Thompson, Esq.—Analyses of Gas—Influence of Atmospheric Pressure and Temperature on Gas—Residual Products—Appendix—Description of Retort Settings, Buildings, etc., etc.

Practical Geometry and Engineering Drawing; a Course of Descriptive Geometry adapted to the Requirements of the Engineering Draughtsman, including the determination of cast shadows and Isometric Projection, each chapter being followed by numerous examples; to which are added rules for Shading Shade-lining, etc., together with practical instructions as to the Lining, Colouring, Printing, and general treatment of Engineering Drawings, with a chapter on drawing Instruments. By GEORGE S. CLARKE, Lieut. R.E., Instructor in Mechanical Drawing, Royal Indian Engineering College. 20 *plates,* 4to, cloth, 15*s.*

The Elements of Graphic Statics. By Professor KARL VON OTT, translated from the German by G. S. CLARKE, Lieut. R.E., Instructor in Mechanical Drawing, Royal Indian Engineering College. *With* 93 *illustrations,* crown 8vo, cloth, 5*s.*

The Principles of Graphic Statics. By GEORGE SYDENHAM CLARKE, Lieut. Royal Engineers. *With* 112 *illustrations.* 4to, cloth, 12*s.* 6*d.*

The New Formula for Mean Velocity of Discharge of Rivers and Canals. By W. R. KUTTER. Translated from articles in the 'Cultur-Ingenieur,' by LOWIS D'A. JACKSON, Assoc. Inst. C.E. 8vo, cloth, 12*s*. 6*d*.

Practical Hydraulics; a Series of Rules and Tables for the use of Engineers, etc., etc. By THOMAS BOX. Fifth edition, *numerous plates*, post 8vo, cloth, 5*s*.

A Practical Treatise on the Construction of Horizontal and Vertical Waterwheels, specially designed for the use of operative mechanics. By WILLIAM CULLEN, Millwright and Engineer. *With* 11 *plates*. Second edition, revised and enlarged, small 4to, cloth, 12*s*. 6*d*.

Aid Book to Engineering Enterprise Abroad. By EWING MATHESON, M. Inst. C.E. The book treats of Public Works and Engineering Enterprises in their inception and preliminary arrangement; of the different modes in which money is provided for their accomplishment; and of the economical and technical considerations by which success or failure is determined. The information necessary to the designs of Engineers is classified, as are also those particulars by which Contractors may estimate the cost of works, and Capitalists the probabilities of profit. *Illustrated*, 2 vols., 8vo, 12*s*. 6*d*. each.

The Essential Elements of Practical Mechanics; based on the Principle of Work, designed for Engineering Students. By OLIVER BYRNE, formerly Professor of Mathematics, College for Civil Engineers. Third edition, *with* 148 *wood engravings*, post 8vo, cloth, 7*s*. 6*d*.

CONTENTS:

Chap. 1. How Work is Measured by a Unit, both with and without reference to a Unit of Time—Chap. 2. The Work of Living Agents, the Influence of Friction, and introduces one of the most beautiful Laws of Motion—Chap. 3. The principles expounded in the first and second chapters are applied to the Motion of Bodies—Chap. 4. The Transmission of Work by simple Machines—Chap. 5. Useful Propositions and Rules.

The Practical Millwright's and Engineer's Ready Reckoner; or Tables for finding the diameter and power of cog-wheels, diameter, weight, and power of shafts, diameter and strength of bolts, etc. By THOMAS DIXON. Fourth edition, 12mo, cloth, 3*s*.

Breweries and Maltings: their Arrangement, Construction, Machinery, and Plant. By G. SCAMELL, F.R.I.B.A. Second edition, revised, enlarged, and partly rewritten. By F. COLYER, M.I.C.E., M.I.M.E. *With* 20 *plates*, 8vo, cloth, 18*s*.

A Practical Treatise on the Manufacture of Starch, Glucose, Starch-Sugar, and Dextrine, based on the German of L. Von Wagner, Professor in the Royal Technical School, Buda Pesth, and other authorities. By JULIUS FRANKEL; edited by ROBERT HUTTER, proprietor of the Philadelphia Starch Works. *With* 58 *illustrations*, 344 pp., 8vo, cloth, 18*s*.

A Practical Treatise on Mill-gearing, Wheels, Shafts, Riggers, etc.; for the use of Engineers. By THOMAS BOX. Third edition, *with* 11 *plates.* Crown 8vo, cloth, 7*s.* 6*d.*

Mining Machinery: a Descriptive Treatise on the Machinery, Tools, and other Appliances used in Mining. By G. G. ANDRÉ, F.G.S., Assoc. Inst. C.E., Mem. of the Society of Engineers. Royal 4to, uniform with the Author's Treatise on Coal Mining, containing 182 *plates*, accurately drawn to scale, with descriptive text, in 2 vols., cloth, 3*l.* 12*s.*

CONTENTS:

Machinery for Prospecting, Excavating, Hauling, and Hoisting—Ventilation—Pumping—Treatment of Mineral Products, including Gold and Silver, Copper, Tin, and Lead, Iron, Coal, Sulphur, China Clay, Brick Earth, etc.

Tables for Setting out Curves for Railways, Canals, Roads, etc., varying from a radius of five chains to three miles. By A. KENNEDY and R. W. HACKWOOD. *Illustrated*, 32mo, cloth, 2*s.* 6*d.*

The Science and Art of the Manufacture of Portland Cement, with observations on some of its constructive applications. *With* 66 *illustrations.* By HENRY REID, C.E., Author of 'A Practical Treatise on Concrete,' etc., etc. 8vo, cloth, 18*s.*

The Draughtsman's Handbook of Plan and Map Drawing; including instructions for the preparation of Engineering, Architectural, and Mechanical Drawings. *With numerous illustrations in the text, and* 33 *plates* (15 *printed in colours*). By G. G. ANDRÉ, F.G.S., Assoc. Inst. C.E. 4to, cloth, 9*s.*

CONTENTS:

The Drawing Office and its Furnishings—Geometrical Problems—Lines, Dots, and their Combinations—Colours, Shading, Lettering, Bordering, and North Points—Scales—Plotting—Civil Engineers' and Surveyors' Plans—Map Drawing—Mechanical and Architectural Drawing—Copying and Reducing Trigonometrical Formulæ, etc., etc.

The Boiler-maker's and Iron Ship-builder's Companion, comprising a series of original and carefully calculated tables, of the utmost utility to persons interested in the iron trades. By JAMES FODEN, author of 'Mechanical Tables,' etc. Second edition revised, *with illustrations*, crown 8vo, cloth, 5*s.*

Rock Blasting: a Practical Treatise on the means employed in Blasting Rocks for Industrial Purposes. By G. G. ANDRÉ, F.G.S., Assoc. Inst. C.E. *With* 56 *illustrations and* 12 *plates*, 8vo, cloth, 10*s.* 6*d.*

Surcharged and different Forms of Retaining Walls. By J. S. TATE. *Illustrated*, 8vo, sewed, 2*s.*

A Treatise on Ropemaking as practised in public and private Rope-yards, with a Description of the Manufacture, Rules, Tables of Weights, etc., adapted to the Trade, Shipping, Mining, Railways, Builders, etc. By R. CHAPMAN, formerly foreman to Messrs. Huddart and Co., Limehouse, and late Master Ropemaker to H.M. Dockyard, Deptford. Second edition, 12mo, cloth, 3*s*.

Laxton's Builders' and Contractors' Tables; for the use of Engineers, Architects, Surveyors, Builders, Land Agents, and others. Bricklayer, containing 22 tables, with nearly 30,000 calculations. 4to, cloth, 5*s*.

Laxton's Builders' and Contractors' Tables. Excavator, Earth, Land, Water, and Gas, containing 53 tables, with nearly 24,000 calculations. 4to, cloth, 5*s*.

Sanitary Engineering: a Guide to the Construction of Works of Sewerage and House Drainage, with Tables for facilitating the calculations of the Engineer. By BALDWIN LATHAM, C.E., M. Inst. C.E., F.G.S., F.M.S., Past-President of the Society of Engineers. Second edition, *with numerous plates and woodcuts*, 8vo, cloth, 1*l*. 10*s*.

Screw Cutting Tables for Engineers and Machinists, giving the values of the different trains of Wheels required to produce Screws of any pitch, calculated by Lord Lindsay, M.P., F.R.S., F.R.A.S., etc. Royal 8vo, cloth, oblong, 2*s*.

Screw Cutting Tables, for the use of Mechanical Engineers, showing the proper arrangement of Wheels for cutting the Threads of Screws of any required pitch, with a Table for making the Universal Gas-pipe Threads and Taps. By W. A. MARTIN, Engineer. Second edition, royal 8vo, oblong, cloth, 1*s*., or sewed, 6*d*.

A Treatise on a Practical Method of Designing Slide-Valve Gears by Simple Geometrical Construction, based upon the principles enunciated in Euclid's Elements, and comprising the various forms of Plain Slide-Valve and Expansion Gearing; together with Stephenson's, Gooch's, and Allan's Link-Motions, as applied either to reversing or to variable expansion combinations. By EDWARD J. COWLING WELCH, Memb. Inst. Mechanical Engineers. Crown 8vo, cloth, 6*s*.

Cleaning and Scouring: a Manual for Dyers, Laundresses, and for Domestic Use. By S. CHRISTOPHER. 18mo, sewed, 6*d*.

A Handbook of House Sanitation; for the use of all persons seeking a Healthy Home. A reprint of those portions of Mr. Bailey-Denton's Lectures on Sanitary Engineering, given before the School of Military Engineering, which related to the "Dwelling," enlarged and revised by his Son, E. F. BAILEY-DENTON, C.E., B.A. *With* 140 *illustrations*, 8vo, cloth, 8*s*. 6*d*.

Treatise on Valve-Gears, with special consideration of the Link-Motions of Locomotive Engines. By Dr. GUSTAV ZEUNER. Third edition, revised and enlarged, translated from the German, with the special permission of the author, by MORITZ MÜLLER. *Plates*, 8vo, cloth, 12*s.* 6*d.*

A Pocket-Book for Boiler Makers and Steam Users, comprising a variety of useful information for Employer and Workman, Government Inspectors, Board of Trade Surveyors, Engineers in charge of Works and Slips, Foremen of Manufactories, and the general Steam-using Public. By MAURICE JOHN SEXTON. Second edition, royal 32mo, roan, gilt edges, 5*s.*

The Strains upon Bridge Girders and Roof Trusses, including the Warren, Lattice, Trellis, Bowstring, and other Forms of Girders, the Curved Roof, and Simple and Compound Trusses. By THOS. CARGILL, C.E.B.A.T., C.D., Assoc. Inst. C.E., Member of the Society of Engineers. *With* 64 *illustrations, drawn and worked out to scale*, 8vo, cloth, 12*s.* 6*d.*

A Practical Treatise on the Steam Engine, containing Plans and Arrangements of Details for Fixed Steam Engines, with Essays on the Principles involved in Design and Construction. By ARTHUR RIGG, Engineer, Member of the Society of Engineers and of the Royal Institution of Great Britain. Demy 4to, *copiously illustrated with woodcuts and* 96 *plates*, in one Volume, half-bound morocco, 2*l.* 2*s.*; or cheaper edition, cloth, 25*s.*

This work is not, in any sense, an elementary treatise, or history of the steam engine, but is intended to describe examples of Fixed Steam Engines without entering into the wide domain of locomotive or marine practice. To this end illustrations will be given of the most recent arrangements of Horizontal, Vertical, Beam, Pumping, Winding, Portable, Semi-portable, Corliss, Allen, Compound, and other similar Engines, by the most eminent Firms in Great Britain and America. The laws relating to the action and precautions to be observed in the construction of the various details, such as Cylinders, Pistons, Piston-rods, Connecting-rods, Cross-heads, Motion-blocks, Eccentrics, Simple, Expansion, Balanced, and Equilibrium Slide-valves, and Valve-gearing will be minutely dealt with. In this connection will be found articles upon the Velocity of Reciprocating Parts and the Mode of Applying the Indicator, Heat and Expansion of Steam Governors, and the like. It is the writer's desire to draw illustrations from every possible source, and give only those rules that present practice deems correct.

Barlow's Tables of Squares, Cubes, Square Roots, Cube Roots, Reciprocals of all Integer Numbers up to 10,000. Post 8vo, cloth, 6*s.*

Camus (M.) Treatise on the Teeth of Wheels, demonstrating the best forms which can be given to them for the purposes of Machinery, such as Mill-work and Clock-work, and the art of finding their numbers. Translated from the French, with details of the present practice of Millwrights, Engine Makers, and other Machinists, by ISAAC HAWKINS. Third edition, *with* 18 *plates*, 8vo, cloth, 5*s.*

A Practical Treatise on the Science of Land and Engineering Surveying, Levelling, Estimating Quantities, etc., with a general description of the several Instruments required for Surveying, Levelling, Plotting, etc. By H. S. MERRETT. Third edition, 41 *plates with illustrations and tables*, royal 8vo, cloth, 12*s*. 6*d*.

PRINCIPAL CONTENTS:

Part 1. Introduction and the Principles of Geometry. Part 2. Land Surveying; comprising General Observations—The Chain—Offsets Surveying by the Chain only—Surveying Hilly Ground—To Survey an Estate or Parish by the Chain only—Surveying with the Theodolite—Mining and Town Surveying—Railroad Surveying—Mapping—Division and Laying out of Land—Observations on Enclosures—Plane Trigonometry. Part 3. Levelling—Simple and Compound Levelling—The Level Book—Parliamentary Plan and Section—Levelling with a Theodolite—Gradients—Wooden Curves—To Lay out a Railway Curve—Setting out Widths. Part 4. Calculating Quantities generally for Estimates—Cuttings and Embankments—Tunnels—Brickwork—Ironwork—Timber Measuring. Part 5. Description and Use of Instruments in Surveying and Plotting—The Improved Dumpy Level—Troughton's Level—The Prismatic Compass—Proportional Compass—Box Sextant—Vernier—Pantagraph—Merrett's Improved Quadrant—Improved Computation Scale—The Diagonal Scale—Straight Edge and Sector. Part 6. Logarithms of Numbers—Logarithmic Sines and Co-Sines, Tangents and Co-Tangents—Natural Sines and Co-Sines—Tables for Earthwork, for Setting out Curves, and for various Calculations, etc., etc., etc.

Saws: the History, Development, Action, Classification, and Comparison of Saws of all kinds. By ROBERT GRIMSHAW. *With* 220 *illustrations*, 4to, cloth, 12*s*. 6*d*.

A Supplement to the above; containing additional practical matter, more especially relating to the forms of Saw Teeth for special material and conditions, and to the behaviour of Saws under particular conditions. *With* 120 *illustrations*, cloth, 9*s*.

A Guide for the Electric Testing of Telegraph Cables. By Capt. V. HOSKIŒR, Royal Danish Engineers. *With illustrations*, second edition, crown 8vo, cloth, 4*s*. 6*d*.

Laying and Repairing Electric Telegraph Cables. By Capt. V. HOSKIŒR, Royal Danish Engineers. Crown 8vo, cloth, 3*s*. 6*d*.

A Pocket-Book of Practical Rules for the Proportions of Modern Engines and Boilers for Land and Marine purposes. By N. P. BURGH. Seventh edition, royal 32mo, roan, 4*s*. 6*d*.

Table of Logarithms of the Natural Numbers, from 1 to 108,000. By CHARLES BABBAGE, Esq., M.A. Stereotyped edition, royal 8vo, cloth, 7*s*. 6*d*.

To ensure the correctness of these Tables of Logarithms, they were compared with Callett's, Vega's, Hutton's, Briggs', Gardiner's, and Taylor's Tables of Logarithms, and carefully read by nine different readers; and further, to remove any possibility of an error remaining, the stereotyped sheets were hung up in the Hall at Cambridge University, and a reward offered to anyone who could find an inaccuracy. So correct are these Tables, that since their first issue in 1827 no error has been discovered.

The Steam Engine considered as a Heat Engine: a Treatise on the Theory of the Steam Engine, illustrated by Diagrams, Tables, and Examples from Practice. By JAS. H. COTTERILL, M.A., F.R.S., Professor of Applied Mechanics in the Royal Naval College. 8vo, cloth, 12*s.* 6*d.*

The Practice of Hand Turning in Wood, Ivory, Shell, etc., with Instructions for Turning such Work in Metal as may be required in the Practice of Turning in Wood, Ivory, etc.; also an Appendix on Ornamental Turning. (A book for beginners.) By FRANCIS CAMPIN. Second edition, *with wood engravings*, crown 8vo, cloth, 6*s.*

CONTENTS:

On Lathes—Turning Tools—Turning Wood—Drilling—Screw Cutting—Miscellaneous Apparatus and Processes—Turning Particular Forms—Staining—Polishing—Spinning Metals—Materials—Ornamental Turning, etc.

Health and Comfort in House Building, or Ventilation with Warm Air by Self-Acting Suction Power, with Review of the mode of Calculating the Draught in Hot-Air Flues, and with some actual Experiments. By J. DRYSDALE, M.D., and J. W. HAYWARD, M.D. Second edition, with Supplement, *with plates*, demy 8vo, cloth, 7*s.* 6*d.*

Treatise on Watchwork, Past and Present. By the Rev. H. L. NELTHROPP, M.A., F.S.A. *With* 32 *illustrations*, crown 8vo, cloth, 6*s.* 6*d.*

CONTENTS:

Definitions of Words and Terms used in Watchwork—Tools—Time—Historical Summary—On Calculations of the Numbers for Wheels and Pinions; their Proportional Sizes, Trains, etc.—Of Dial Wheels, or Motion Work—Length of Time of Going without Winding up—The Verge—The Horizontal—The Duplex—The Lever—The Chronometer—Repeating Watches—Keyless Watches—The Pendulum, or Spiral Spring—Compensation—Jewelling of Pivot Holes—Clerkenwell—Fallacies of the Trade—Incapacity of Workmen—How to Choose and Use a Watch, etc.

Spons' Engineers' and Contractors' Illustrated Book of Prices of Machines, Tools, Ironwork, and Contractors' Material; and Engineers' Directory. Third edition, 4to, cloth, 6*s.*

Algebra Self-Taught. By W. P. HIGGS, M.A., D.Sc., LL.D., Assoc. Inst. C.E., Author of 'A Handbook of the Differential Calculus,' etc. Second edition, crown 8vo, cloth, 2*s.* 6*d.*

CONTENTS:

Symbols and the Signs of Operation—The Equation and the Unknown Quantity—Positive and Negative Quantities—Multiplication—Involution—Exponents—Negative Exponents—Roots, and the Use of Exponents as Logarithms—Logarithms—Tables of Logarithms and Proportionate Parts—Transformation of System of Logarithms—Common Uses of Common Logarithms—Compound Multiplication and the Binomial Theorem—Division, Fractions, and Ratio—Continued Proportion—The Series and the Summation of the Series—Limit of Series—Square and Cube Roots—Equations—List of Formulæ, etc.

JUST PUBLISHED.

In super-royal 8vo, 1168 pp., *with* 2400 *illustrations*, in 3 Divisions, cloth, price 13*s*. 6*d*. each; or 1 vol., cloth, 2*l*.; or half-morocco, 2*l*. 8*s*.

A SUPPLEMENT

TO

SPONS' DICTIONARY OF ENGINEERING,

Civil, Mechanical, Military, and Naval.

EDITED BY ERNEST SPON, MEMB. SOC. ENGINEERS.

THE success which has attended the publication of 'SPONS' DICTIONARY OF ENGINEERING' has encouraged the Publishers to use every effort tending to keep the work up to the standard of existing professional knowledge. As the Book has now been some years before the public without addition or revision, there are many subjects of importance which, of necessity, are either not included in its pages, or have been treated somewhat less fully than their present importance demands. With the object, therefore, of remedying these omissions, this Supplement is now being issued. Each subject in it is treated in a thoroughly comprehensive way; but, of course, without repeating the information already included in the body of the work.

The new matter comprises articles upon

Abacus, Counters, Speed Indicators, and Slide Rule.
Agricultural Implements and Machinery.
Air Compressors.
Animal Charcoal Machinery.
Antimony.
Axles and Axle-boxes.
Barn Machinery.
Belts and Belting.
Blasting. Boilers.
Brakes.
Brick Machinery.
Bridges.
Cages for Mines.
Calculus, Differential and Integral.
Canals.
Carpentry.
Cast Iron.
Cement, Concrete, Limes, and Mortar.
Chimney Shafts.
Coal Cleansing and Washing.
Coal Mining.
Coal Cutting Machines.
Coke Ovens. Copper.
Docks. Drainage.
Dredging Machinery.
Dynamo - Electric and Magneto-Electric Machines.
Dynamometers.
Electrical Engineering, Telegraphy, Electric Lighting and its practical details, Telephones
Engines, Varieties of.
Explosives. Fans.
Founding, Moulding and the practical work of the Foundry.
Gas, Manufacture of.
Hammers, Steam and other Power.
Heat. Horse Power.
Hydraulics.
Hydro-geology.
Indicators. Iron.
Lifts, Hoists, and Elevators.
Lighthouses, Buoys, and Beacons.
Machine Tools.
Materials of Construction.
Meters.
Ores, Machinery and Processes employed to Dress.
Piers.
Pile Driving.
Pneumatic Transmission.
Pumps.
Pyrometers.
Road Locomotives.
Rock Drills.
Rolling Stock.
Sanitary Engineering.
Shafting.
Steel.
Steam Navvy.
Stone Machinery.
Tramways.
Well Sinking.

NOW COMPLETE.

With nearly 1500 *illustrations*, in super-royal 8vo, in 5 Divisions, cloth.
Divisions 1 to 4, 13*s*. 6*d*. each; Division 5, 17*s*. 6*d*.; or 2 vols., cloth, £3 10*s*.

SPONS' ENCYCLOPÆDIA

OF THE

INDUSTRIAL ARTS, MANUFACTURES, AND COMMERCIAL PRODUCTS.

EDITED BY C. G. WARNFORD LOCK, F.L.S.

Among the more important of the subjects treated of, are the following:—

Acids, 207 pp. 220 figs.
Alcohol, 23 pp. 16 figs.
Alcoholic Liquors, 13 pp.
Alkalies, 89 pp. 78 figs.
Alloys. Alum.
Asphalt. Assaying.
Beverages, 89 pp. 29 figs.
Blacks.
Bleaching Powder, 15 pp.
Bleaching, 51 pp. 48 figs.
Candles, 18 pp. 9 figs.
Carbon Bisulphide.
Celluloid, 9 pp.
Cements. Clay.
Coal-tar Products, 44 pp. 14 figs.
Cocoa, 8 pp.
Coffee, 32 pp. 13 figs.
Cork, 8 pp. 17 figs.
Cotton Manufactures, 62 pp. 57 figs.
Drugs, 38 pp.
Dyeing and Calico Printing, 28 pp. 9 figs.
Dyestuffs, 16 pp.
Electro-Metallurgy, 13 pp.
Explosives, 22 pp. 33 figs.
Feathers.
Fibrous Substances, 92 pp. 79 figs.
Floor-cloth, 16 pp. 21 figs.
Food Preservation, 8 pp.
Fruit, 8 pp.
Fur, 5 pp.
Gas, Coal, 8 pp.
Gems.
Glass, 45 pp. 77 figs.
Graphite, 7 pp.
Hair, 7 pp.
Hair Manufactures.
Hats, 26 pp. 26 figs.
Honey. Hops.
Horn.
Ice, 10 pp. 14 figs.
Indiarubber Manufactures, 23 pp. 17 figs.
Ink, 17 pp.
Ivory.
Jute Manufactures, 11 pp., 11 figs.
Knitted Fabrics — Hosiery, 15 pp. 13 figs.
Lace, 13 pp. 9 figs.
Leather, 28 pp. 31 figs.
Linen Manufactures, 16 pp. 6 figs.
Manures, 21 pp. 30 figs.
Matches, 17 pp. 38 figs.
Mordants, 13 pp.
Narcotics, 47 pp.
Nuts, 10 pp.
Oils and Fatty Substances, 125 pp.
Paint.
Paper, 26 pp. 23 figs.
Paraffin, 8 pp. 6 figs.
Pearl and Coral, 8 pp.
Perfumes, 10 pp.
Photography, 13 pp. 20 figs.
Pigments, 9 pp. 6 figs.
Pottery, 46 pp. 57 figs.
Printing and Engraving, 20 pp. 8 figs.
Rags.
Resinous and Gummy Substances, 75 pp. 16 figs.
Rope, 16 pp. 17 figs.
Salt, 31 pp. 23 figs.
Silk, 8 pp.
Silk Manufactures, 9 pp. 11 figs.
Skins, 5 pp.
Small Wares, 4 pp.
Soap and Glycerine, 39 pp. 45 figs.
Spices, 16 pp.
Sponge, 5 pp.
Starch, 9 pp. 10 figs.
Sugar, 155 pp. 134 figs.
Sulphur.
Tannin, 18 pp.
Tea, 12 pp.
Timber, 13 pp.
Varnish, 15 pp.
Vinegar, 5 pp.
Wax, 5 pp.
Wool, 2 pp.
Woollen Manufactures, 58 pp. 39 figs.

London: E. & F. N. SPON, 16, Charing Cross.
New York: 35, Murray Street.